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INVESTIGATION OF STATISTICAL NATURE OF FATIGUE PROPERTIES

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TABLE OF CONTENTS

	Page
SUMMARY	1
INTRODUCTION	1
LITERATURE REVIEW	3
Statistics of Fatigue-Fracture Curve	3
Statistics of Endurance Limit	8
Inclusions	9
I. ROTATING-CANTILEVER FATIGUE TESTS	10
Experimental Work	10
Material and preparation of fatigue specimens	10
Test procedure	11
Calculation of stress	12
Experimental Results and Analysis	14
Statistics of fatigue-fracture curve	14
Statistics of endurance limit	18
Statistics of location of crack initiation	20
Discussion	23
Statistical nature of fatigue properties	23
S-N diagram	24
II. VIBRATING-CANTILEVER FATIGUE TESTS	31
Experimental Work	31
Materials and preparation of fatigue specimens	31
Test procedure	34
Calculation of stress	35
Experimental Results and Analysis	45
Statistics of fatigue-fracture curve	47
Statistics of endurance limit	50
S-N diagram	53
Understressing study	54
Discussion	57
Dependence of statistical variation in fatigue life on stress level	58
Influence of metallurgical factors on statistical behavior	60
Influence of experimental error	62
Theories of failure and statistics	64
Size effect	66
S-N diagram	68
Understressing effect	69
Engineering application	73

Page

CONCLUSIONS 74

REFERENCES 76

TABLE 80

FIGURES 83

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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By E. Epremian and R. F. Mehl

SUMMARY

A thorough review of the literature indicated the need for research on the statistical nature of the fatigue of metals. While much effort has been directed toward the effect of numerous variables on the average fatigue properties, the statistical variation of the mean properties and its causes have been virtually neglected. An extensive experimental program was undertaken to study this subject and to determine and evaluate the fundamental factors which influence this behavior.

The statistics of the fatigue-fracture curves and endurance limits were determined for a variety of materials and, by analysis of these experimental results, the effects of some metallurgical factors on the statistical nature of fatigue properties were shown. It was discovered that inclusions play a dominant rôle in this behavior and that other factors such as composition and microstructure are of secondary importance.

In addition, a number of other aspects of the problem were studied, namely, the dependence of statistical variations in fatigue life on stress level in the fracture range, the statistics of the location of crack initiation, the size effect and the understressing effect from a statistical viewpoint, and the form of the S-N diagram and the method of plotting it. For the most part, the experimental work was done in the pneumatic vibratory fatigue test machine which is more suitable than the usual R. R. Moore fatigue machine for these investigations.

INTRODUCTION

Of the many ways in which metals can fail under the application of stresses, fatigue is probably the most widespread and important. Any mechanical part which undergoes vibration, rotation, or reciprocation can suffer fatigue failure, and it has been reported that 95 percent of service failures in machine parts are by fatigue. Thus the problem of fatigue is of utmost importance to the design engineer who must properly understand and recognize the phenomena involved in this type of failure in order to design equipment and machinery for successful operation.

The fatigue of metals is of great scientific interest as well as practical importance. This peculiar type of failure occurs at stresses which are far less than those which produce fracture in the ordinary manner such as under a tensile load. This fact and many other fatigue phenomena require theoretical interpretation including a fundamental mechanism which explains how and why this type of failure occurs.

Before the present investigation of the statistical nature of fatigue properties was undertaken, a thorough critical review was made of all the literature pertaining to the fatigue of metals. This study included the experimental knowledge of the subject as well as the theories which have been proposed to explain the observed phenomena. As a result of this study, it was evident that many aspects of this field are in need of more research, particularly the statistical nature of fatigue which has been virtually neglected. Further, many conclusions drawn from the usual investigations can be questioned since the statistical approach, now known to be necessary, was not used.

Fatigue is demonstrably a statistical phenomenon, but the fundamental factors which influence this behavior have not been understood. It has been known for sometime that the fatigue life of a metal at a given stress varies statistically and more recently it was discovered, in this laboratory, that the endurance limit is also a statistical quantity and not an exact value. Thus both the fracture (finite life) and endurance ranges are subject to statistical variation and there is a finite probability of premature failures in the fracture range and of the occurrence of failure below the so-called endurance limit. The dependence of this behavior on metallurgical factors was not known and it was toward this objective that much of the experimental work was directed.

Dispersion in fatigue data can arise from three principal sources: The test machines, the preparation of specimens, and the metal itself. Poor alinement, calibration, and operation of the fatigue machines and lack of control of specimen preparation (surface finish, etc.) would obviously produce scatter in the test results. It will be shown, however, that even when the experimental variables are properly controlled the fatigue life and endurance limit are statistical in nature, and <u>this variability is inherent in the material</u>.

It was thought that the statistical variation of fatigue properties depends upon the cleanliness, composition, strength level, and microstructure of the metal. For example, a material with a large number of inclusions might have a greater statistical variation in fatigue properties than a material of similar composition and microstructure with very few inclusions. Or, for a given composition and inclusion rating, the variability might depend upon the type of microstructure, and so on.

At Carnegie Institute of Technology, under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics, a program was designed to study the statistical nature of fatigue and to evaluate the fundamental factors that affect the variability. Extensive fatigue tests were made on annealed Armco iron and plain carbon and alloy steels heat-treated to different strengths and microstructures. The statistics of the fatigue-fracture curves and endurance limits were determined; by analysis of these experimental results, the effects of metallurgical factors on the statistical nature of fatigue properties were shown. Only the longitudinal direction has been considered in this work.

In addition, a number of other aspects of the problem were considered, namely, study of the dependence of statistical variation in fatigue life on stress level in the fracture range, study of the statistics of the location of crack initiation, study of the size effect and the understressing effect from a statistical viewpoint, and study of the form of the S-N diagram and the methods of plotting it.

For convenience, the presentation of the experimental work is made in two parts since two types of test machines were used in the study. In part I, experiments were conducted on normalized and tempered SAE 1050 steel in rotating-cantilever machines of the R. R. Moore type. In part II, annealed Armco iron and SAE 4340 steel heat-treated to different strengths and microstructures were fatigue tested in pneumatic vibrating-cantilever machines.

LITERATURE REVIEW

A complete survey of all aspects of the subject of the fatigue of metals was made to determine what work had been done on the subject (both experimental and theoretical) and to recognize the important variables involved, appraise their relative influence, and control them in experiment. A detailed account of this survey would be much too lengthy for inclusion here; for the present purposes, a thorough review and analysis of the existing evidence for the statistical behavior of fatigue will be sufficient.

Statistics of Fatigue-Fracture Curve

Scatter is found in practically every set of fatigue data and it becomes increasingly evident in the S-N diagram with increase in the number of specimens tested. There have been a number of experimental investigations in which the scatter in fatigue life is clearly detectable.

Bollenrath and Cornelius (reference 1) conducted fatigue tests in the fracture range on aluminum and magnesium alloys, steels, copper, and brass and obtained considerable scatter in the data. However, the number of stress levels investigated and the number of specimens tested per level were limited so that evaluation of the results by statistical methods is difficult. Müller-Stock (reference 2) tested 200 specimens of a steel at one stress level in the fracture range and found a frequency distribution in the results. As seen in figure 1, this distribution is skew with respect to N (the number of cycles to failure), but if this distribution is determined on the basis of $\log N$ it is found that it becomes normal. Freudenthal, Yen, and Sinclair (reference 3) studied SAE 1045 and SAE 4340 steels and tough pitch copper by testing 20 specimens at each of a few levels of stress in the fracture range. Similar tests were made by Yen and Dolan (reference 4) on 75S-T alloy. That portion of the results of these two investigations which is of interest for the present purposes is summarized in the following table:

Material (1)	Stress (psi)	Mean of logs of lives, $\overline{\log N}$	Standard deviation, σ	Mean life, $\overline{N}$ (cycles)	Approximate endurance limit (psi)
SAE 1045	67,500	4.625	0.104	4.2×10^4	48,000
	61,000	5.061	.103	1.2×10^5	
	55,000	5.633	.241	4.3	
SAE 4340	118,000	4.897	0.166	7.9×10^4	85,000
	105,000	5.261	.225	1.8×10^5	
Copper	22,500	5.382	0.037	2.4×10^3	
75S-T	62,500	4.204	0.036	1.6×10^4	
	50,000	5.000	.070	1.0×10^5	
	37,500	6.357	.193	2.3×10^6	

[1]Specimens were heat-treated as follows: SAE 1045, normalized from 1600° F; SAE 4340, oil-quenched from 1525° F and tempered $1\frac{1}{2}$ hours at 1000° F; copper (tough pitch), annealed 1 hour at 700° F; 75S-T, precipitation-hardening treatment not given.

From a superficial examination of the values shown in this table, it appears that the standard deviations σ for the two steels differ only slightly and therefore these materials have essentially the same statistical behavior in fatigue failure. Closer examination, however, plainly shows that the alloy steel has greater variability than the plain carbon steel. In order to compare the σ values properly, it is necessary to convert the data to a common basis. This is done by recalculating the stresses as percentages of the endurance limits and then comparing the variance of the two steels at equivalent stress levels. The application of statistical methods to these data will be made in a later section, where detailed analysis of all existing data for the variance of fatigue properties will be given. By the use of these methods, it is possible to determine whether differences in variability of the fatigue properties of various materials are significant.

Peterson (reference 5) has recently proposed an approximate statistical method for the analysis of fatigue data. An average S-N curve is drawn through the points such that the sum of the differences (in stress) from the average is 0. The percentage difference in stress of each experimental point from the average curve is calculated and transferred along the S-N curve to a chosen point of longer life (and thus lower stress). This procedure provides a distribution of stress values at a chosen fatigue life which can be analyzed in the usual manner for the mean and standard deviation. This method applies only to the fatigue-fracture curve and involves the assumption that the statistics at high stresses can be extrapolated to low stresses. This assumption is unjustified (as will be proven later), but the approach may be useful in obtaining approximate results. The following table lists some results obtained by Peterson (unpublished data). It should be pointed out that this method yields a mean value of stress and a standard deviation at a designated number of cycles, whereas the usual methods give a mean number of cycles and a standard deviation at a given stress. The advantage of Peterson's method is, of course, that an estimate of the variability in the fatigue life of a metal can be obtained with much fewer tests than are required by the usual method.

Material	Experimenter	σ (psi)	V (percent) (1)
SAE 1050	H. F. Moore	2070	3.1
SAE 1050 (notched)	H. F. Moore	1428	4.3
SAE 1045	O. Horger	1180	3.5
SAE 3140	T. J. Dolan and C. S. Yen	1520	2.3

[1] $V = 100 \frac{\sigma}{\overline{X}}$ where σ is standard deviation and $\overline{X}$ is mean stress at a chosen number of cycles (i.e., V is relative standard deviation). Relative standard deviation is of limited importance since it cannot be used in well-established methods of analysis of differences in variability.

Unfortunately, the heat treatments of these steels were not reported and, since in this method the differences in results obtained at different stress levels relative to the endurance limits are ignored, comparison and interpretation of the results are difficult. Presumably, however, Moore's tests on notched and unnotched specimens were made on the same steel and the results suggest that the presence of a notch increases the variability, but, in view of the fact that this method is an approximation, it is not possible to draw any conclusions with certainty.

Perhaps the most fruitful source of information on the scatter obtained in the fatigue life of metals is the research by Ravilly (reference 6) who conducted torsional fatigue tests on wires of iron, steel, nickel, copper, aluminum, zinc, and silver. With each of these metals, 20 tests were conducted at each of 10 stress levels. All of the tests were conducted at stresses in the fatigue-failure range and therefore all of the specimens broke and the endurance limits were not determined. Ravilly was interested primarily in the mean life for S-N curves and neglected to calculate the statistical parameters of dispersion for the results of his 1000 or more tests. Fortunately, the original data are included in the report so that it has been possible to perform the statistical calculations. As a matter of fact, it was necessary to recalculate the mean-life value at each stress level since Ravilly determined the arithmetic mean of the N values whereas the true mean should be calculated from the $\log N$ values, since the distribution is normal for $\log N$. Since the results of Ravilly's work are of extreme interest with regard to the statistical nature of fatigue properties, they will be reviewed in detail.

The first series of torsional fatigue experiments was conducted on steel wire in both the cold-worked and annealed conditions and, although the chemical composition of the steel, the degree of cold-work, and the annealing treatment are not stated, the results indicate an interesting effect as will be shown. The marked scatter in the data from the first set of experiments led Ravilly to seek a method of presorting the specimens to decrease the dispersion obtained in fatigue tests. In his second series of experiments, Ravilly measured the magnetic differences between the specimens and a chosen standard specimen in a magnetic-comparator apparatus before conducting the fatigue tests. It was found that those specimens which showed the greatest magnetic difference relative to the standard also yielded the greatest deviation in fatigue life from the standard.

Ravilly then conducted extensive tests on annealed specimens of Armco iron, steel, and nickel. The specimens of these materials used for fatigue tests were preselected by the magnetic-comparator method. These results are given in table I since they are too lengthy to be included here. Similar fatigue tests were conducted on annealed copper, aluminum, zinc, and silver (results for the copper and aluminum are also given in table I) and in these cases the materials were not presorted since they are nonmagnetic. Unfortunately, only the means rather than all of the original data were given for zinc and silver so that calculation of the dispersion of the fatigue life of these materials could not be made.

For each of the metals tested there was no consistent trend in the degree of dispersion in fatigue life as a function of the level of stress. In a later section it will be shown by use of mathematical tests that the dispersion in fatigue life in these torsional tests is independent of the stress level in the fracture range. Therefore, an over-all average of the variabilities obtained at the various stress levels for a given material can be taken for visual comparison and the results of the calculations of Ravilly's voluminous data can be summarized in a single table, as follows:

Material (1)	Average σ	Average V (percent)	Total specimens involved
Annealed steel (not presorted)	0.1851	3.482	40
Cold-worked steel (not presorted)	.3896	7.381	40
Annealed Armco iron (not presorted)	.1540	2.820	60
Annealed Armco iron (presorted)	.0283	.569	200
Annealed nickel (presorted)	.0390	.760	200
Annealed steel (presorted)	.0417	.798	140
Annealed copper (not presorted)	.1124	2.296	200
Annealed aluminum (not presorted)	.1218	2.589	200

[1]Chemical compositions, thermal treatment, etc. are not given.

From the information in this table, it is possible to draw a number of conclusions. The validity of these conclusions must be established by analysis of variance as will be done subsequently.

(1) Clearly the statistical variation in the fatigue life of the steel in the cold-worked condition is significantly larger than that obtained in the annealed condition. Apparently inhomogeneities have a greater effect in the cold-worked metal, or perhaps the cold-working operation in itself increases the nonuniformity of the material.

(2) The variability in fatigue life of annealed iron is less than that for annealed steel with both unsorted and preselected specimens.

This result is to be expected since it is likely that the statistical variation in fatigue properties depends upon the cleanliness of the metal. Further evidence on this point is the fact that a limited number of fatigue tests on pure ferrites which were prepared with extreme care showed little variation (reference 7), while data for commercial steels containing inclusions, alloy segregate, and so forth show a wider scatter.

(3) It is also evident from the values in the table that preselection of specimens by the magnetic comparator reduced the dispersion markedly. It is difficult to understand why there should be a correlation between the difference in magnetic response and the deviation in fatigue life from a standard specimen. Unfortunately, Ravilly reported only the magnitude of the differences in magnetic behavior relative to the standard for the various specimens without stating the sign of these differences. Thus it is not possible to determine whether those specimens which had relatively lower magnetic properties had correspondingly lower-than-standard fatigue life.

(4) Comparing the values for unsorted specimens, it is seen that the scatter for the nonferrous materials (copper and aluminum) is less than that for iron or steel. It is regrettable that the purity and inclusion rating of the metals and their annealing treatments are not known since this information would be of extreme value in the interpretation of the results. In any event, it is probable that there are fewer inclusions and inhomogeneities in the former materials and those which are present are less effective because of the high ductility of the copper and aluminum.

Statistics of Endurance Limit

The work reviewed above has dealt solely with the fatigue-fracture curve and until recently there have been no studies of the variability of the endurance limit. Ransom and Mehl (references 8 and 9) investigated the statistical behavior of the fatigue properties of quenched and tempered SAE 4340 and discovered that the endurance limit is markedly statistical in nature. They studied the longitudinal and transverse fatigue properties of two heats of SAE 4340; one lot of material had a high reduction in area in transverse tensile test (high RAT) while the other had a low value (low RAT). A sufficient number of fatigue tests were conducted to permit statistical analysis of the data yielding measures of the dispersion in both the fracture curves and the endurance limits.

It was found as a result of extensive testing that there is a mean endurance limit $\overline{S}$ at which 50 percent of the specimens fail and

50 percent "run out" and a standard deviation σ about this mean endurance limit. For example, in one case (transverse test on low RAT material), $\overline{S}$ was 46,270 psi and σ was 2900 psi. Thus the range $\overline{S} \pm 2\sigma$ which includes 95 percent of the endurance range extends from 40,470 to 52,070 psi, a condition which was previously unknown and unsuspected.

It was concluded in the investigation that the transverse fatigue strength is lower than that in the longitudinal direction and that the statistical variation in the fatigue properties in the transverse direction of the low RAT material is significantly greater than that for the longitudinal direction of the low RAT steel and for both directions in the high RAT steel.

Ransom's data need not be presented here in detail since these results will be included in the final comparison of data for many materials including the results of the present investigation.

Inclusions

While very little is known about the effect of metallurgical factors on the statistical nature of fatigue properties, it is probable that inclusions play a role of some importance. Thus it would be worth while to review briefly the known effects of this variable. The effect of inclusions on the average fatigue properties has been studied, but the influence of different inclusion ratings on the statistical variation in fatigue properties has not been evaluated.

All inclusions are not detrimental to the fatigue properties. Stewart and Williams (reference 10) classified inclusions and determined the effect of various types on the endurance limit of steel. They reported that the presence of angular and elongated inclusions decreases the fatigue properties, while small rounded inclusions do not significantly affect the endurance limit. Dolan and Price (reference 11) report that the addition of 0.2 percent lead (which forms small rounded insoluble particles) to SAE 1045 in the cold-drawn, normalized, or quenched and tempered condition does not impair the fatigue strength. Von Rajakovics (reference 12) found that finely distributed lead particles have no effect on the fatigue limit of the duralumin alloy, while the presence of hard angular constituents reduces the fatigue limit. Ransom reports (in a private communication), as a result of metallographic studies, that fatigue cracks in steel almost always initiate at an inclusion.

These experimental results are not at all surprising since they are in agreement with the existing theories of fatigue and fracture. Fatigue is known to be a localized phenomenon and Orowan (reference 13) has developed a theory of fatigue based on the fact that metals contain imperfections which cause stress concentrations. He points out that under static loading these stress concentrations are reduced by plastic flow; however, in the fatigue process, successive cycles work-harden the material in the neighborhood of the imperfections, thereby continually diminishing the reduction in stress concentration due to plastic flow. This process continues until a condition is approached in which there is no plastic flow either because the yield point is sufficiently increased or because the ultimate strength of the material is exceeded; in the latter case, a fatigue failure occurs.

From this view it is to be expected that inclusions markedly affect fatigue properties. Further, in keeping with Zener's theory of fracture (reference 14), an elongated inclusion should be more effective than a rounded particle in initiating fracture.

I. ROTATING-CANTILEVER FATIGUE TESTS

Experimental Work

Material and preparation of fatigue specimens.- The source of fatigue specimens for this portion of the investigation was a forged 9.91-inch-diameter railroad crankpin of SAE 1050 (Assoc. of Am. Railroads spec. M-10437, class A, forging No. A-11) with the following chemical composition in percent by weight:

C	Mn	P	S	Si	Cr	Ni	Mo
0.49	0.75	0.027	0.031	0.31	0.02	0.15	0.01

The original crankpin was given a normalized and tempered heat treatment by heating up to 1480° F in 7 hours, holding for 11 hours, and removing from the furnace to air-cool, followed by heating up to 1160° F in 6 hours, holding for 10 hours, and furnace cooling.

All 200 fatigue specimens were taken longitudinally from the same 9.25-inch-diameter location of the crankpin, and each was marked on its end to indicate its orientation with respect to the original crankpin, as illustrated in figure 2. The specimens were machined to 0.020-inch oversize on a lathe with a round-nose tool to minimize machine marks, after which they were ground longitudinally to the desired diameter. The specimens were then packed in cast-iron chips and stress-relief annealed by heating for 2 hours at 1100° F followed by furnace cooling. The gage lengths were then polished circumferentially with fine emery paper to remove any grinding marks and surface oxide and finally honed longitudinally. The honing was done in a machine (shown in fig. 3) which rotates the specimen at 1/4 rpm while the polishing stone covered with finishing compound reciprocates longitudinally 180 times per minute, thus producing an extremely fine surface. Before testing, the surface of each specimen was inspected for circumferential scratches under a binocular microscope at 60X and bars were rehoned if necessary. The dimensions of the test specimen are shown in figure 4. Specimen diameters were measured optically to eliminate the possibility of producing micrometer marks on the surface.

The surface finish on the test specimens was measured by a circumferential trace with a Brush Surface Analyzer. Measurements on four bars chosen at random gave the same result of 40 microinches. This value is the distance from the highest peak to the lowest valley on the surface and represents the maximum possible roughness of the surface since the trace was made in a direction perpendicular to the final longitudinal honing.

Examination of the microstructures of longitudinal and transverse sections of fatigue specimens showed no differences. The structure consisted of a proeutectoid-ferrite matrix with patches of pearlite in which some of the cementite was partially spheroidized. An inclusion rating of the material by the standard SAE method gave an index of $3^{d} - 1.4^{300d} - C$. The A.S.T.M. grain size was from 7 to 8. The microstructure is shown in figure 5.

The usual mechanical properties were measured since these data are of comparative and supplementary value in a fatigue study. The following table lists the results of these tests:

Position in crankpin	Yield point (psi)	Ultimate strength (psi)	Elongation (percent)	Reduction in area (percent)	Izod impact (ft-lb)	Rockwell hardness
O.D.	46,500	91,375	24.5	43.7	9,11,9	B-81
Midsection	48,500	91,500	24.5	39.9	9,7,8	B-82

The tensile data are the result of single tests and thus little significance should be placed on the differences between the midsection and outer diameter.

Test procedure.- The fatigue tests were conducted in four rotating machines of the R. R. Moore type at a speed of 8000 rpm. Extreme care was taken in the calibration and adjustment of the equipment to assure good operation. In addition, it was found necessary to replace the original spring method of loading by the use of actual weights to obtain the desired accuracy. One of the fatigue machines with the revised method of loading is pictured in figure 6.

In starting a fatigue test, the bar was allowed to develop its full speed of rotation before the load was gradually applied by adding the proper amount of weights (scrap roller bearings) to the pails. The usual microswitch and counter arrangement was used to measure the number of cycles required for fatigue failure (complete fracture). Specimens which ran to 5×10^7 cycles were considered to be nonfailures.

Approximately 20 specimens were tested at each of nine levels of stress covering both the fracture and endurance ranges. Measurements were made on each fractured specimen to determine the location of crack initiation with respect to the gage length and circumference of the specimen, and in those cases where fracture occurred away from the minimum diameter of the gage length the proper correction in the value of stress was made by use of the curve shown in figure 7. The location of the crack nucleus with respect to the specimen circumference was also recorded using the clock notation shown in figure 2 which is based on the position of the specimen end marks. Location was reported as clock positions 0 to 6, where position 0 lies toward the outer surface of the crankpin and 6 lies toward the core. It was not possible to distinguish between positions 3 and 9 since there are two halves of a broken specimen. Consequently those failures reported for position 3 included those which occurred in position 9. Similarly those reported for position 2 include position 10, position 5 includes position 7, and so forth.

Calculation of stress.- The stress was calculated by the formula $S = \frac{M}{Z}$, where S is the maximum fiber stress in psi, M is the bending moment in inch-pounds, and Z is the section modulus in inches cubed. The bending moment M is given by $M = PL$, where P is the total load in pounds and L is the lever arm in inches. The section modulus which is the moment of inertia divided by the distance from the neutral axis to the outermost fiber is for the present case $Z = \frac{\pi}{32} D^3$, where D is the diameter of the specimen.

The stress is thus a function of the load, the lever arm, and the specimen diameter. Since these quantities are measured and therefore subject to an uncertainty, a corresponding error can be introduced in the calculated stress because of the uncertainty in the measurements of these quantities as given by the expression:

$$dS = \left(\frac{\partial S}{\partial P}\right) dP + \left(\frac{\partial S}{\partial L}\right) dL + \left(\frac{\partial S}{\partial D}\right) dD$$

where $\frac{\partial S}{\partial P}$, $\frac{\partial S}{\partial L}$, and $\frac{\partial S}{\partial D}$ are partial differentials and dP, dL, and dD are exact differentials which represent the uncertainty in measurement.

The percentage error in stress is simply $100 \frac{dS}{S}$ so that the equation can be rewritten as:

$$\begin{aligned} \text{Percentage error in stress} &= 100 \frac{dS}{S} \\ &= \frac{100}{S}\left[\left(\frac{\partial S}{\partial P}\right) dP + \left(\frac{\partial S}{\partial L}\right) dL + \left(\frac{\partial S}{\partial D}\right) dD\right] \end{aligned}$$

The values of $\frac{\partial S}{\partial P}$, $\frac{\partial S}{\partial L}$, and $\frac{\partial S}{\partial D}$ can readily be obtained by taking the partial derivatives of the equation for stress. Thus,

$$S = \frac{M}{Z} = \frac{PL}{Z} = \frac{PL}{\frac{\pi}{32} D^3}$$

therefore $$\left(\frac{\partial S}{\partial P}\right) = \frac{L}{Z}, \quad \left(\frac{\partial S}{\partial L}\right) = \frac{P}{Z}, \quad \text{and} \quad \left(\frac{\partial S}{\partial D}\right) = -\frac{3M}{\frac{\pi}{32} D^2},$$

and

$$\text{Percentage error in stress} = \frac{100}{S}\left[\left(\frac{L}{Z}\right) dP + \left(\frac{P}{Z}\right) dL + \left(-\frac{3M}{\frac{\pi}{32} D^2}\right) dD\right]$$

Then substituting $\frac{PL}{\frac{\pi}{32} D^3}$ for S in the denominator of the right-hand side of the equation yields:

$$\text{Percentage error in stress} = 100\left[\frac{dP}{P} + \frac{dL}{L} + (-3D)\ dD\right]$$

The uncertainty dP in the measurement of the load is ±0.03 pound, while dL is ±0.005 inch and dD is ±0.0001 inch. Thus, for a load of 27 pounds with a 4-inch lever arm and a 0.300-inch-diameter specimen, the equation gives the following result:

$$\text{Percentage error in stress} = 100(\pm 0.00111 \pm 0.00125 \pm 0.00009)$$

$$= \pm 0.25$$

It is evident that the possible error in stress introduced by uncertainties in measured quantities is negligible. The calculated stress with its maximum possible deviation for this example which is typical of the tests conducted is 40,742 ± 102 psi.

Experimental Results and Analysis

The scatter obtained in both the fracture and endurance ranges of stress was appreciable and, since meticulous care was taken in performing the experiments, this scatter was attributed to the metal. Statistical tests for significant differences in the results obtained from the four machines were negative. The broad band within which the results of 200 tests fell is shown in figure 8. It is seen that the highest stress at which a nonfailure was obtained is 42,500 psi, while failures occurred at stresses as low as 40,000 psi. This variability is a direct indication that an exact single value of the endurance limit cannot be stated, but that this quantity must be represented statistically. Similarly, the marked spread in the data for the fatigue-fracture curve suggests the necessity for statistical evaluation of the finite life values. While the broad band drawn in figure 8 gives the limits within which the experimental results lie, a statistical analysis of the distribution of the results within the limits is far more informative and useful.

Statistics of fatigue-fracture curve.- Calculations for the fatigue life were made for each of the stress levels studied in the fracture range using statistical methods (references 15 and 16). The observed numbers of cycles to fracture for the various tests at a given stress were compiled in tables from which the statistical parameters were computed.

Of particular interest is the arithmetic mean which represents the point at which the data "pile up." This average value is obtained by dividing the sum of the measurements by the number of observations. Thus:

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

where $\overline{X}$ is the arithmetic mean, X_i is a measurement, and n is the number of observations. In the present work, the variable involved is N, the number of cycles to fracture at a given stress. However, since it is known that the distribution is normal with respect to log N rather than N itself, the former is used in the calculation of the mean. Thus the expression for the arithmic mean becomes:

$$\overline{\log N} = \frac{\sum_{i=1}^{n} \log N_i}{n}$$

where $\overline{\log N}$ is the mean of the logarithms of the fatigue-life values. Log N_i is the logarithm of a measurement of the number of cycles to fracture and n is the number of tests conducted at a given stress level. The mean fatigue life in terms of the actual number of cycles $\overline{N}$ is obtained by taking the antilogarithm of $\overline{\log N}$.[1] The results of calculations of the mean fatigue life at the various stress levels are summarized in the following table:

Stress (psi)	Mean of logs of life values, $\overline{\log N}$	Mean life, $\overline{N}$ (cycles)	Unbiased standard deviation, σ	Standard estimate of error, $\sigma_{\overline{\log N}}$	Relative standard deviation, V (percent)
45,000	5.73548	5.438×10^5	0.1458	0.03436	2.542
44,000	5.97802	9.507	.2126	.04638	3.556
43,000	6.17060	1.481×10^6	.1601	.03773	2.594
42,500	6.15482	1.429	.2212	.04946	3.593
42,000	6.28467	1.926	.2696	.06740	4.289
41,500	6.34349	2.205	.1768	.04903	2.787
41,000	6.46617	2.925	.2030	.05425	3.139

[1]The value of $\overline{N}$ is not to be construed as the arithmetic mean of the number of cycles to failure, but simply as the antilogarithm of $\overline{\log N}$. However, it will be convenient to use $\overline{N}$ when talking about the expected range and distribution in fatigue life.

It is seen that the mean fatigue life increases with decrease in stress as expected, but it will be shown that the variation in fatigue life with stress in an S-N diagram does not follow the simple curve which is usually drawn.

While the arithmetic mean represents that value at which the results tend to "pile up," it does not indicate the dispersion in the data. A convenient measure of the dispersion in a set of observations is σ, the standard deviation. For a normal distribution, the range of $\pm 1\sigma$ about the mean includes 68 percent of the observations, the $\pm 2\sigma$ range includes 95 percent, and the $\pm 3\sigma$ range includes 99.7 percent of the results. The formula with which the standard deviation is calculated from the data is:

$$\sigma = \left(\frac{\sum_{i=1}^{n} x_i^2}{n} - \overline{X}^2 \right)^{1/2}$$

or for the present case:

$$\sigma = \left[\frac{\sum_{i=1}^{n} (\log N)^2}{n} - \left(\overline{\log N}\right)^2 \right]^{1/2}$$

using the same nomenclature as before.

It is recognized that a group of test specimens constitutes a sample of all possible specimens in a parent universe. Thus, because of the sample size, the value calculated for a group of specimens by the above equation may not apply to the universe. An unbiased value of the standard deviation for the universe is obtained by multiplying the value for the sample by $\left(\frac{n}{n - 1}\right)^{1/2}$ where n is the sample size. It is evident that this factor becomes significant when the sample size is small.

Another statistical parameter of interest is the standard estimate of error which is a measure of the possible error between the arithmetic mean of a group of specimens and the true mean of the universe. This quantity is calculated by the formula:

$$\sigma_{\overline{\log N}} = \frac{\sigma}{\sqrt{n}}$$

where $\sigma_{\overline{\log N}}$ is the standard estimate of error, σ is the unbiased standard deviation, and n is the sample size or number of specimens in the group. The significance of the standard estimate of error is that there is a 68-percent probability that the true mean for the universe lies within the range of $\pm 1\sigma_{\overline{\log N}}$ about the sample mean $\left(\text{or a 95-percent probability for the } \pm 2\sigma_{\overline{\log N}} \text{ range, etc.}\right)$.

Values of the unbiased standard deviation and the standard estimate of error are listed in the preceding table and plotted with the mean life and stress in figures 9 and 10. Figure 9 shows the mean life and the ranges of $\pm 1\sigma_{\overline{\log N}}$ and $\pm 2\sigma_{\overline{\log N}}$ about the mean for the various levels of stress. It appears likely that the fatigue-fracture curve is not a straight line or a simple curve concave upward to the right, as it is usually drawn. While such simple curves may be drawn through the data in figure 9 by selecting values within the 95-percent band $\left(\pm 2\sigma_{\overline{\log N}}\right)$, use of the mean or the 68-percent band $\left(\pm 1\sigma_{\overline{\log N}}\right)$ indicates that the fatigue curve has a point of inflection and, being asymtotic to the mean endurance limit and bending toward the stress axis, has a form somewhat like that of a sigmoid curve.

The statistical distribution of fatigue life about the mean is shown in figure 10 from which it is evident that there is appreciable dispersion. For example, at a stress of 45,000 psi, the most probable fatigue life (the mean) is 5.2×10^5 cycles while there is a 68-percent probability ($\pm 1\sigma$) of obtaining a life between 3.8×10^5 and 7.6×10^5 cycles, or a 95-percent probability ($\pm 2\sigma$) of obtaining a fatigue life between 2.7×10^5 and 1.0×10^6 cycles.

A question of extreme interest and importance arises with regard to the statistical distribution of the fatigue life: Does the variability in life change in a consistent manner with change in the level of stress in the fracture range? It has been the general belief that the dispersion in fatigue life increases with decrease in stress in the fracture range. An answer to the question might be obtained by visual inspection of the spread (σ values) in the results plotted in figure 10, but this analysis is somewhat superficial and a rigorous mathematical treatment would be more desirable. Such a treatment is available (reference 17) and its application to the problem has shown that within the range of stresses used, the variability in fatigue life did not depend upon the stress level. It should be emphasized, however, that the range studied was through necessity quite narrow. The method of the mathematical test used is briefly summarized.

A parameter M known as Hartley's statistic is calculated from the data and compared with M values given in a table. If the obtained value of M is less than that given in the table, it may be concluded that the differences in the values corresponding to the various stress levels is not significant. The Hartley statistic is calculated from the equation:

$$M = N\left[\log_e\left(\frac{\sum_{t=1}^{K} \nu_t\sigma_t^{\,2}}{N}\right)\right] - \left(\sum_{t=1}^{K} \nu_t \log_e \sigma_t^{\,2}\right)$$

where

$\sigma_t^{\,2}$ variance

$\nu_t = N_t - 1$

N_t number of specimens tested at given stress level

$t = 1, 2, \ldots K$

K number of stress levels investigated

$N = \nu_1 + \nu_2 + \ldots + \nu_k$

The value of M obtained from the data was 9.53 which is lower than the value of 12.59 from the table, and it may therefore be concluded that the variance dispersion is not significant within the limited range of stresses investigated.

Similar calculations of the Hartley statistic were made with the SAE 1045 and 75S-T data, taken from the first table presented in the section "Literature Review," which cover a wider stress range. In both cases, it was found that the variance dispersion is significant so it may be said that here the dispersion in fatigue life does increase with decrease in stress level within the fracture range.

Statistics of endurance limit.- In making a statistical analysis of the endurance limit the actual number of cycles a specimen undergoes is not of importance as it is in the study of the fracture curve. In the endurance range the consideration is whether the specimen breaks or does not break and it is assumed that if a life of 5×10^7 cycles is obtained without fracture, the specimen may be considered a nonfailure. The situation is similar to toxic tests on bugs in biological research. Groups of bugs are subjected to various lethal dosages and the proportion of deaths noted. The number of minutes a bug lives after being poisoned is of no interest, but the percentage of bugs which succumb to given dosages yields useful information when analyzed statistically. In fatigue tests, the level of stress corresponds to the concentration of the dosage in toxic tests and fatigue failure represents death. Thus, the highly developed statistical methods used in biological research (reference 18) can readily be applied to study of the endurance limit in fatigue.

The percentage of failures for specimens of SAE 1050 obtained at the various stress levels are given in the following table:

Stress (psi)	Number of failures	Total tests	Percent failures
43,000	18	18	100.0
42,500	20	21	95.2
42,000	16	18	88.9
41,500	13	20	65.0
41,000	14	21	66.7
40,500	8	21	38.1
40,000	4	17	23.5

As expected, the proportion of failures increased with increase in stress level. A "mortality" curve is plotted on linear coordinates in figure 11 using the data given in the preceding table and it is seen that the values fall on a sigmoid curve. It will be recalled that a sigmoid curve can be expressed analytically by the probability integral or Gaussian error function and thus becomes a straight line when plotted on probability graph paper. Figure 12 shows the data replotted on probability paper yielding a good straight line.

The stress $\overline{S}$ at which 50 percent of the nonfailures occurred (defined as the mean endurance limit) was 40,800 psi. The standard deviation σ is the reciprocal of the slope of the straight line in figure 12 and is obtained from the equation:

$$Y = 5 + \frac{1}{\sigma}\left(S - \overline{S}\right)$$

where

σ	standard deviation
$\bar{S}$	mean endurance limit, psi
S	level of stress, psi
Y	probit of percentage of failures corresponding to stress S
5	probit of 50 percent of failures

The probit is a solution of the Gaussian error function for a given percentage. Convenient tables are available (reference 18) for changing percentages to probits (probability graph paper makes a somewhat similar transformation).[2]

Calculation of σ yields a value of 1028 psi. Thus, $\bar{S} \pm 2\sigma = 40,800 \pm 2056$ psi and the range 4σ (which includes 95 percent of the possible endurance limits) is 4112 psi. The mean endurance limit and its standard deviation are shown in figure 10. It is seen that the endurance and fracture ranges overlap and the question might arise as to why nonfailures were not obtained at stresses higher than 42,500 psi (the highest observed stress for a nonfailure) as predicted by the statistics. For example, the statistics predict that at a stress of 43,940 psi (3140 psi above the mean endurance limit) there should be 0.1-percent nonfailures; but, in order to get one nonfailure at this stress, it is necessary to test 1000 specimens, while only 20 specimens were tested at each stress level. Similarly, the statistics predict that one out of every 1000 specimens tested at a stress of 37,660 psi (3140 psi below the mean endurance limit) would break.

Statistics of location of crack initiation.- Measurements of the location of fatigue crack initiation with respect to both the gage length and circumference of the specimens were analyzed statistically. This portion of the work is of interest for two reasons. If failure initiates at a point along the gage length away from the minimum diameter, it implies that the stress at this point (although nominally

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[2]This statement of the probit method is simplified for convenience. The actual procedure involves drawing a provisional line through the data points, a test of the goodness of fit of this line with the experimental points by a χ^2 test, and a mathematical solution for the mean and slope rather than a simple graphical solution. The method is presented in detail by Finney (reference 18).

lower) is actually as high as or higher than the stress at the minimum diameter, presumably because of the stress concentrating effect of some internal inhomogeneity. Since extreme care was taken in the preparation of specimens and the calibration of the machines, it is felt that such behavior could not be attributed to experimental error. Thus, the frequency with which failures occur at points away from the minimum section is an indication of the inhomogeneous nature of the material. Secondly, comparison of the number of failures which occur at positions around the specimen circumference with the probable number predicted for random behavior gives an indication of the degree of homogeneity of the original crankpin from which the specimens were obtained.

The distance along the specimen gage length from the point at which fracture initiated to the center line (point of minimum diameter) was measured and reported as a section number. For example, those fractures which occurred within 1/64 inch from the minimum diameter were considered to be in section 1, those which occurred between -1/64 to -2/64 inch and between 1/64 and 2/64 inch from the minimum diameter were reported as being in section 2, and so forth. The location of the sections with respect to the specimen is shown in figure 13. The number of failures which occurred in the various sections is summarized in the following table and plotted in figure 14. It is shown that a considerable

Section	Number of failures	Percent failures
1	54	38.6
2	35	25.0
3	19	13.6
4	16	11.4
5	6	4.29
6	3	2.14
7	3	2.14
8	2	1.43
9	1	.714
10	1	.714

number of fractures initiate at points away from the minimum diameter and that this number decreases with increase in distance from the center line. The curve in figure 14 appears similar to one type of probability curve and therefore suggests the possibility of obtaining a straight line on probability graph paper. The data from the preceding table, replotted on probability paper in figure 15, approximate a straight line fairly well. This result is further evidence of the statistical aspect of the behavior of metals under alternating stresses.

The frequency with which fracture occurs at points removed from the minimum section would depend upon the radius of curvature of the specimen gage length as well as the size and number of inhomogeneities. Certainly, if sharply V-notched specimens were used, there would be no failures initiated at points other than the minimum section. No attempt was made to calculate a distribution of stress concentrators from the the data in figures 7 and 14, but the existence of such a statistical distribution can be inferred from the results.

The results of observations of the location of crack initiation with respect to the circumference are summarized below. A certain

Clock position	Number of failures	Fraction of failures	Theoretical fraction for randomness	Percentage deviation from randomness
0	9	0.06428	0.0833	-22.93
1	21	.1500	.1666	-9.96
2	15	.1071	.1666	-35.71
3	26	.1857	.1666	11.46
4	23	.1643	.1666	-1.38
5	25	.1786	.1666	7.20
6	21	.1500	.0833	80.07

fraction of the total number of failures occurred at each of the clock positions and comparison of this fraction with the theoretical fraction based on complete randomness leads to an interesting result.

On the basis of randomness, there is an equal probability of fracture initiation at each of the 12 clock positions (0 to 11) around the circumference. Thus the theoretical fraction of failures for each of 12 positions is 1/12 or 0.0833. As mentioned previously, measurement cannot distinguish between positions 3 and 9, 2 and 10, 5 and 7, and so forth, while positions 0 and 6 are unique. Therefore, the theoretical fraction for each of the positions 1 through 5, inclusive, (which include positions 11 through 7, respectively) is 2 times 0.0833 or 0.1666 and is 0.0833 for positions 0 and 6. The percentage deviations of the observed from the theoretical values based on randomness are given in the preceding table and plotted in figure 16. No great significance is placed in the actual numerical values, but the marked tendency is obvious. Failure initiated far more often in locations which lie toward the core of the original crankpin than would be predicted on the basis of probabilities for random behavior.

While the actual forging reduction from ingot to crankpin is not known, it probably can be assumed that this reduction was no more than 4:1 (the crankpin diameter is 9.91 in.). This reduction is insufficient for thorough hot-work of the ingot to its core, which might explain why failures tended to initiate at specimen locations which lie toward the center of the original crankpin.

Discussion

Statistical nature of fatigue properties.- There can be no doubt that fatigue properties are statistical in nature. This fact is extremely important in engineering design with regard to both the fracture and endurance ranges. In the design of parts for a finite length of service, such as automobile parts, it must be recognized that statistically some failures may occur prematurely. For example, it has been found that at a stress of 45,000 psi the mean life of the SAE 1050 steel is 5.2×10^5 cycles, but there is a 16-percent probability of having a life less than 3.8×10^5 cycles and a 2.5-percent probability of obtaining failure in less than 2.7×10^5 cycles. Such variability in the fatigue life should be recognized and taken into consideration in the choice of the proper factor of safety. Similarly, the statistical nature of the endurance limit should be a source of concern to the engineer in designing parts for infinite life. It has been shown that 0.1-percent failures would occur, in the material studied, at a stress of 3140 psi below the mean endurance limit. Information of this type for the material involved should be a guide to the designer in his selection of operating stresses.

The interesting comparison of the results of this statistical study of the fatigue life and endurance limit with the data for other metals will be deferred until after the experimental work of part II has been presented.

While this portion of the investigation did not give a very satisfactory answer to the question, other work has indicated that the dispersion in fatigue life increases with decrease in stress level within the fracture range. This conclusion is still subject to some doubt, however, because all the work which shows this effect has been done in rotating test machines which define fatigue failure by complete fracture. Thus, the number of cycles measured includes the cycles required to propagate a crack as well as those necessary to initiate failure. Bennett (reference 19) has shown that the rate of propagation of a fatigue crack increases rapidly with increase in the level of stress, so that propagation is much more rapid at high stresses than at low levels. These differences in the rate of crack propagation are included in the measured scatter of the fatigue life and may in part explain why the dispersion increases with decrease in stress level. In part II of the experimental work, this problem is studied properly by using test machines which define fatigue failure by the initiation of a crack rather than complete fracture, thus eliminating the complication of the rate of crack propagation.

S-N diagram.- One of the interesting results of this investigation is the observation that the fatigue-fracture curve in the S-N diagram is not a straight line or a simple curve, concave upward to the right as it is usually drawn. It has been found that the curve has a point of inflection, bending toward the stress axis at shorter life (higher stress) and toward the mean endurance limit at longer life. This result may readily be understood; indeed it is in agreement with prediction.

The tensile strength represents the stress required to produce fracture in one-quarter of an alternating cycle (i.e., from zero to maximum stress). If the tensile strength is plotted in an S-N diagram, it is seen at once that the curve must bend toward the stress axis at shorter life in the diagram. Rigorously, use should not be made of the tensile strength under static loading but of the value obtained at the same strain rate as that of the fatigue test. The correction for strain rate in the tensile strength is small, however, so that the dynamic tensile strength (at the same strain rate as a fatigue test) is approximately 1.25 times the static tensile strength (reference 20). The general form of the predicted S-N curve is not changed appreciably.

To determine the shape of the complete S-N diagram experimentally it would be necessary to conduct tests at a number of relatively high stresses approaching the tensile strength. One of the principal difficulties encountered in fatigue tests at very high stresses is the development of an adiabatic condition. Heating of specimens due to the relatively high stresses and frequency of alternation complicate the experiments, but perhaps this effect could be overcome by using coolants and a lower speed of test. Experiments at these high stresses are included in part II.

Recently a paper by Wallgren (reference 21) has been published in which appear several complete S-N diagrams. One of these is reproduced in figure 17 to show the form of the curve which is in agreement with the prediction made above. It is seen that the S-N curve has a point of inflection and extends backward continuously to the tensile strength. Wallgren has used the static tensile strength and plotted its value at 1 cycle, which is essentially the same as one-quarter of a cycle. No mention is made of the heating effect during test, but presumably it was avoided since some of the experiments were conducted at frequencies as low as 180 cycles per minute.

The importance of knowing the proper form of the S-N curve cannot be overemphasized. The present method of plotting fatigue data, which is essentially the same as used by Wöhler in 1865, yields no information beyond what the experimental points themselves show. However, if the true form of the curve is known, the proper coordinates may be chosen so that a straight-line fatigue plot will be obtained. The advantages of a straight-line relationship between stress and number of cycles to failure are apparent. Such a relationship permits calculation of parameters which are characteristic of the material, facilitates extrapolation and interpolation of the data, and serves as a check on spurious data points.

Weibull (reference 22) has attempted to obtain a straight-line S-N plot by assuming the following equation:

$$N = k(S - \overline{S})^{-m}$$

where

N	number of cycles to failure
k,m	constants
S	stress, psi
$\overline{S}$	endurance limit, psi

Thus, according to this view, a plot of $\log N$ against $\log (S - \overline{S})$ is a straight line. Using Ravilly's data, Weibull has shown that such a plot gives either a straight line or two straight lines. It is quite obvious, however, that this equation cannot apply to the entire S-N curve since extrapolation to $N = \frac{1}{4}$ gives a value which is many times larger than the tensile strength. Weibull's equation has been applied to the data on SAE 1050 obtained in this investigation and the data obtained by Wallgren and in both cases the straight-line relationship was not obtained.

It is proposed that a new equation based on the Gaussian function expresses the relationship between stress and the mean number of cycles to failure more adequately. An application of the proposed method is illustrated in figure 18, utilizing Wallgren's data. The stresses are converted to a percentage of the range between the endurance limit and the tensile strength and plotted against $\log N$ on probability paper to obtain a good straight line. The mathematical relationships involved and the usefulness of the method will now be demonstrated.

The normal statistical distribution curve is given by the Gaussian error function:

$$y = \phi(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} e^{-t^2/2}\, dt$$

where

ϕ error function of X

X function of number of cycles to failure

t variable of integration

This curve has the form of a sigmoid curve. The reciprocal of this curve is simply:

$$y = \left[1 - \phi(X)\right]$$

The complete S-N curve extends continuously from the tensile strength to the endurance, as shown in figure 17 taken from Wallgren's work (reference 21). If the stress values are recalculated on a percentage basis of the range between the mean endurance limit and the tensile strength, then the S-N curve has the same form as the curve for $y = \left[1 - \phi(X)\right]$. This percent stress is given by:

$$S^* = \frac{S - \overline{S}}{S_u - \overline{S}}\, 100$$

where

S^* percent stress (percentage of range between mean endurance limit and tensile strength)

S stress, psi

$\overline{S}$ mean endurance limit, psi

S_u tensile strength (preferably corrected for strain rate), psi

Thus the equation for the fatigue curve becomes $S^* = 100\left[1 - \phi(X)\right]$ and the parameter X is a function of the number of cycles to failure. This parameter is taken as $X = a \log N - c$, where a and c are constants for the material. The constant a has a value of 1 if the curve has the same slope as the Gaussian function (normal) in probability coordinates and the constant c provides for the necessary displacement of the curve with respect to the abscissa (it will be recalled that the abscissa of the Gaussian function extends from minus to plus values while, of course, the fatigue curve has all positive values).

Having made these substitutions, the equation for the fatigue curve becomes:

$$S^* = 100\left[1 - \phi(X)\right]$$

where

$$X = a \log N - c$$

$$\phi(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} e^{-X^2/2}\, dx$$

$$S^* = \left(\frac{S - \overline{S}}{S_u - \overline{S}}\right)100$$

It is this function (i.e., S^* against $\log N$) which gives a straight line when plotted on probability paper (see fig. 18) with slope a and intercept c.

Substituting the expression for S^* back in the equation yields:

$$\left(\frac{S - \overline{S}}{S_u - \overline{S}}\right)100 = 100\left[1 - \phi(a \log N - c)\right]$$

and solving for S gives:

$$S = \overline{S}\left\{1 + \left(\frac{S_u}{\overline{S}} - 1\right)\left[1 - \phi(a \log N - c)\right]\right\}$$

which is an expression for the stress as a function of the number of cycles to fracture involving the endurance limit, the tensile strength, and the two constants. For the special case of when the endurance-limit - tensile-strength ratio is 1/2, the equation becomes:

$$S = \bar{S}\left[2 - \phi(a \log N - c)\right]$$

The use of these relationships in the representation of fatigue data will be illustrated by an example using results from Wallgren's work. All stresses are calculated as percentages by means of the equation $S^* = \left(\frac{S - \bar{S}}{S_u - \bar{S}}\right)100$. Having determined S^* values, the corresponding values of $\phi(a \log N - c)$ can be computed by using the relation $S^* = 100\left[1 - \phi(a \log N - c)\right]$. Once the values of $\phi(a \log N - c)$ are known, the corresponding quantities of $(a \log N - c)$ can be obtained from probability error function tables and finally the constants a and c can be evaluated.

For example, it is known that the tensile strength is 106 kilograms per square millimeter and the mean endurance limit is 20 kilograms per square millimeter for the steel tested by Wallgren. It was found that at a stress of 23.4 kilograms per square millimeter the fatigue life is 10^6 cycles. Thus when $\log N$ is 6 (i.e., $\log 10^6 = 6$):

$$S^* = \left(\frac{S - \bar{S}}{S_u - \bar{S}}\right)100 = \frac{23.4 - 20}{106 - 20} = 4 \text{ percent}$$

Similarly other S^* and $\log N$ combinations may be calculated from the data. The corresponding value of $\phi(a \log N - c)$ is calculated as follows:

$$S^* = 100\left[1 - \phi(a \log N - c)\right]$$

$$4 = 100\left[1 - \phi(a \log N - c)\right]$$

therefore

$$\phi(a \log N - c) = 0.96$$

From probability error function tables, it is known that when $\phi(X) = 0.96$ then $X = 1.8$. Thus $(a \log N - c) = 1.8$ and, since in this case $\log N = 6$, the equation $(6a - c) = 1.8$ is obtained. Repeating this procedure for several combinations of S* and log N results in a series of similar equations, any two of which can be used to evaluate a and c. The following table summarizes a few values calculated from Wallgren's data.

S* (percent)	log N	$\phi(a \log N - c)$	$(a \log N - c)$
4	6	0.96	1.8
13	5	.87	1.1
30	4	.70	.5
54	3	.46	-.1
76	2	.24	-.7

Using simultaneous equations, for example:

$$(5a - c) = 1.1$$

$$(3a - c) = -0.1$$

it is found that $a = 0.6$ and $c = 1.9$. Substituting these constants back in the general equations yields:

$$S^* = 100\left[1 - \phi(0.6 \log N - 1.9)\right]$$

or

$$S = 20\left\{1 + 4.3\left[1 - \phi(0.6 \log N - 1.9)\right]\right\}$$

These expressions quantitatively describe the relationship between the stress and the number of cycles to fracture which has hitherto been unknown. It is expected that the constants a and c depend upon the material, but the general method should apply in all cases.

Having established this relationship for one case, it would be desirable to apply the same treatment to other fatigue data. Unfortunately, the other data in the literature are not suitable for this purpose. Most fatigue data, including the experimental results on SAE 1050 reported herein, do not cover a sufficiently wide range of stresses to provide a complete plot by this method. The deficiency is particularly marked at higher stresses where failure occurs in a number of cycles on the order of 10^4. Then, too, most determinations of the S-N curve are made with too few specimens to establish accurately the mean life at each stress level, which must be known in order to check properly the validity of the proposed relationship. One of the purposes of the experiments reported in part II is to obtain the necessarily complete data for checking the proposed method of presenting fatigue data.

There is another point with regard to the S-N diagram that is worthy of careful consideration. It has generally been believed for many years that only iron and steels have endurance limits and that other materials have no safe range under fatigue stresses. This myth will probably be as difficult to dispel as the old concept that fatigue failure occurs because the metal has "crystallized."

There is ample evidence that many nonferrous materials do have an endurance limit and undoubtedly a safe range can be found for all materials provided the tests are continued long enough and at sufficiently low stresses. Gough (reference 23) determined the endurance limit of single crystals of pure aluminum. Naturally, the stress was low, but the existence of a safe range cannot be disputed; at the endurance-limit stress, slip occurred during the earlier stages of test but ceased after some number of cycles whereupon an elastic state was achieved and failure was not obtained even beyond 10^8 cycles. This behavior is exactly the same as that observed in endurance-limit tests on ferrous materials.

A review of the literature discloses the fact that endurance limits have been found for other nonferrous materials as well. Fatigue data for pure titanium (reference 24) given in figure 19, for pure lead (reference 25) given in figure 20, and for pure gold (reference 26) given in figure 21 show the existence of endurance limits for these materials. Thus, there is no longer any reason to believe that only ferrous materials have a safe range. Indeed, from the existing knowledge of the mechanism of fatigue, perhaps it is to be expected that ferrous and nonferrous materials do not differ in their fundamental behavior under cyclic stresses.

II. VIBRATING-CANTILEVER FATIGUE TESTS

Experimental Work

Materials and preparation of fatigue specimens.- The Armco iron was obtained in the form of a hot-rolled plate (3/4 by 39 in., sheared into 7-in. lengths) with the following chemical composition in percent by weight:

C	Mn	S	Cu	P	Si
0.020	0.027	0.019	0.069	0.003	0.001

The original plate which had an A.S.T.M. grain size of 4 was heat-treated to refine and equiax the grains by heating to 650° C in $3\frac{1}{2}$ hours, holding for $15\frac{1}{2}$ hours at temperature, and furnace cooling. This long-time anneal was used because the large mass of the Armco plate prevented rapid cooling to a subcritical temperature as required for a gamma-alpha-type heat treatment for refining the grain structure. The resultant structure from the annealing treatment (shown in fig. 22) had an A.S.T.M. grain size of 6. The SAE inclusion rating was 2^{vd} - 1.1^{9vd} - D. All of the Armco iron fatigue specimens were taken longitudinally from the heat-treated plate and their orientation was such that their plane of bending in flexural test was perpendicular to the plane of rolling. The specimens were machined and finish-ground to the dimensions indicated in figure 23 and given a final stress-relief anneal in a vacuum furnace by heating at 500° C for 1 hour and slow cooling. The specimens were given a final longitudinal polish with 3/0 emery paper and carefully inspected for surface imperfections before testing.

Hardness and tensile tests were made, the results of which are given below. The values reported are the means of three determinations.

Ultimate strength, psi	42,200
0.1-percent yield strength, psi	19,900
Elongation, percent	42.9
Reduction in area, percent	75.9
Rockwell hardness	B-53

Hot-rolled bar stock of SAE 4340 was obtained from an aluminum-killed, basic electric-furnace heat with the following chemical composition in percent by weight:

C	Mn	P	S	Si	Cr	Ni	Mo
0.35	0.83	---	---	0.27	0.77	1.82	0.37

This composition is almost identical to that of the steel used by Ransom in his research (reference 8).

A number of preliminary heat-treatment experiments were made on this material to establish its suitability with regard to quench-cracking susceptibility, since it was found that other heats of this type of steel cracked badly upon quenching pieces with the dimensions required for the fatigue specimens. Fatigue specimen blanks $\left(7/16 \text{ by } 13/16 \text{ by } 6\frac{1}{2} \text{ in.}\right)$ were machined from this material and several other heats of SAE 4340 for which the quench-cracking susceptibilities were known. These blanks were subjected to oil- and water-quenching after an austenitizing treatment (2 hr at 845° C) and then sectioned, macro-etched, and carefully examined for quench cracks. The results of these experiments correlated very nicely with the work done at this laboratory on a standard quench-cracking susceptibility test (references 27 to 29). It was found that fatigue specimen blanks of those steels which had a poor quench-cracking index in the standard susceptibility test (e.g., a 3/16-in. index) cracked upon quenching in either water or oil. On the other hand, those heats of SAE 4340 which had a favorable index in the standard test (index greater than 1/2 in.) did not crack when quenched in either oil or water in the form of fatigue specimen blanks. The heat taken for the present fatigue experiments was one of the latter and more desirable group. Further attempts to produce quench cracks in this steel by repeated austenitizing-quenching treatments were unsuccessful, so that the material was considered to be suitable in this respect.

The bulk of the origin.l bar stock (9/16- by 1-in. flat) was then cut into 7-inch lengths, rough-machined to 7/16 by 13/16 inch to remove the mill scale and decarburized layer, and then heat-treated. Fatigue specimens were then machined, ground, polished, and inspected in the same manner as were the Armco iron specimens. The stress-relief annealing treatment was, in this case, at a temperature of 50° C below the tempering temperature and the thickness of the specimen was 0.0250 inch. One set of fatigue specimens of SAE 4340 was given a quenched and tempered heat treatment which corresponds to a tensile strength of approximately 160,000 psi, while another set was quenched and spheroidized to a tensile strength of 100,000 psi. These treatments were chosen to cover a range in strength levels with different microstructures and to facilitate comparison of the results with Ransom's work on the statistical nature of the fatigue properties of SAE 4340, heat-treated to a tensile strength of 125,000 psi. The proper times and temperatures for heat treatment to develop the desired strengths and structures were obtained after an extensive study of the microstructures, tensile properties, and hardnesses produced by different tempering treatments. This steel had an SAE inclusion rating of $4^{vd} - 1.7^{28vd} - B$. The microstructures obtained from the two heat treatments used are shown in figures 24 and 25 and the corresponding tensile strengths and hardnesses are summarized in the following table:

Heat treatment	Ultimate strength (psi)	0.1-percent yield strength (psi)	Elongation (percent)	Reduction in area (percent)	Rockwell hardness
Quenched and tempered[1]	167,790	153,662	17.6	55.2	C-38
Quenched and spheroidized[2]	103,482	83,120	27.5	62.2	B-98

[1]2 hr at 845° C - oil-quenched; 16 hr at 575° C - oil-quenched.
[2]2 hr at 845° C - oil-quenched; 16 hr at 600° C - oil-quenched.

The tensile data given are the averages of three tests, while the hardness values are averages of 125 measurements.

Extensive hardness tests were made to determine the degree of uniformity of the heat treatment of SAE 4340. The quenching of the specimens of this steel was done in batches and it was feared that this procedure might have produced some variation in hardness either across the section of a given specimen or from specimen to specimen. Therefore, 25 specimens were taken at random from each of the two gross lots of heat-treated fatigue-test bar blanks, sectioned at a distance of 1/2 to 1 inch from the end, and tested for hardness. The measurements were made at points adjacent to each of the four sides of the cross section and at the center. It was found that there was no consistent difference in hardness at the various points on the cross section in either lot of specimens. The fact that the center hardnesses were equivalent to surface hardnesses indicated uniform quenching throughout. The maximum difference in Rockwell hardness in a given specimen was C-2.4 for the quenched and tempered specimen and B-1.9 in the other, while the average was C-1.5 and B-1.1, respectively. These values establish the uniformity in hardness within individual specimens. As to the uniformity from specimen to specimen it was found that the unbiased standard deviation from the mean Rockwell hardness of the 25 specimens was C-1.0 in the case of the quenched and tempered specimens and B-0.80 in the case of the quenched and spheroidized specimens. From these data, it was concluded that the heat-treatment procedure did not impose a variation in the mechanical properties of the specimens.

Test procedure.- The fatigue tests were conducted in eight General Electric Company pneumatic vibrating-cantilever machines. This type of fatigue test machine was designed by Quinlan and has been described in his publications (references 30 and 31) as well as the A.S.T.M. "Manual on Fatigue Testing" (reference 32). The machine pneumatically actuates a cantilever specimen at its natural frequency of vibration. Air jets are directed at pistons mounted on the free end of the specimen and the length of the pneumatic column is adjusted so that its resonant frequency coincides with that of the specimen. The amplitude of vibration of the specimen is controlled by the amount of air pressure and flow and is measured visually by means of a graduated eyepiece in a small telescope mounted on the side of the machine. The frequency of vibration, which depends upon the physical constants and dimensions of the specimen, is detected by a coil-magnet arrangement in conjunction with a frequency meter and is recorded on a slip chart. This machine and some of its control equipment is shown in figure 26.

Fatigue failure in the usual rotating-beam machine is defined by complete fracture of the specimen and, as pointed out in a previous section, this criterion is inaccurate since it includes not only the number of cycles to initiate failure but also those required to propagate the crack. The Quinlan pneumatic machine, on the other hand, is capable of giving a true measurement of fatigue failure, that is, the number of cycles to incipient failure. A sample tested in the pneumatic cantilever machine vibrates at its natural frequency and maintains this constant value during the test until the last stage when a minute crack forms. At this stage of the test (point A in fig. 27), the frequency of vibration declines because of the initiation of a crack and finally the specimen ceases to vibrate when the crack becomes quite large (point B). The fatigue machines and their controls have been designed and adjusted so that the test is automatically shut off when the frequency of vibration decreases below the initial value, thus indicating the occurrence of true failure.

The control mechanism is such that when the pointer which indicates the frequency of vibration drops below the natural value contact is made with a positioned terminal, whereupon an electronic relay and a magnetic switch turn off the power in the system, thus stopping the time recorder and shutting off the air supply by deenergizing a solenoid valve in the airline. The number of cycles to incipient failure is obtained by multiplying the frequency of vibration by the time of test. As will be shown, the stress in a given test is readily determined from measurements of the frequency and amplitude of vibration.

Tests were conducted at a number of stress levels selected to cover the entire fracture and endurance ranges. At each stress level, approximately 20 specimens were tested as had been done in the work reported in part I.

Ironically, one of the more serious experimental difficulties encountered in this phase of the work was due to fatigue itself. Normally the fatigue cracks form at the fillet of the vibrating-cantilever specimens (see fig. 23). Initially, however, a number of the specimens, particularly in the lower stress tests, broke below the fillet in the clamped portion of the specimen. These peculiar failures were due to fret-corrosion fatigue and were not included in the data reported. This difficulty was eliminated by shot-peening the portions of the specimens which were clamped in the holder on the machine.

Calculation of stress.- There are a number of methods by which the stress in the cantilever fatigue specimens can be evaluated and it becomes a question of adopting that procedure which is most convenient and accurate. One possible method is the use of dynamic strain gages to obtain strain measurements at various amplitudes of the vibrating cantilever beam. It is found that strain varies linearly with the amplitude of vibration and if the modulus of elasticity is known, the stress can readily be calculated by the formula:

$$S = kAE$$

where

S	stress, psi
k	constant determined from plot of slope of strain against amplitude
A	amplitude, inches
E	modulus of elasticity, psi

This method can be used but it has the disadvantage of necessitating a separate determination of the modulus of elasticity.

This difficulty can be avoided by the introduction of a second relationship:

$$f = c\sqrt{\frac{E}{\rho}}$$

where

f	natural frequency of flexural vibration, cycles per second
c	constant depending upon dimensions of specimen
E	modulus of elasticity, psi
ρ	density of material, pounds per cubic inch

If this equation is combined with the previous one, the following formula is obtained

$$S = KAf^2$$

where K is a new constant related to the previous ones. This equation is more convenient since it eliminates the need for Young's modulus and involves the frequency of vibration which is easily measured during the fatigue test. The problem of determining the constant in this case, however, is more complex. The constant K is related to the mass and dimensions of the specimen, the mass and location of the attached pistons, and the deflection curve for the specimen-piston assembly. A complete analytical expression for the stress equation has been derived and a numerical solution made for the conditions which exist in the tests made in this investigation. The mathematical treatment by which the convenient stress equation was obtained will now be given.

Derivation: The derivation is based on the formula:

$$S = \frac{M}{Z}$$

where

S	stress, psi
M	total bending moment, inch-pounds
Z	section modulus, inches^3

The nature of harmonic motion is considered to obtain an expression for the forces acting on the specimen. The total bending moment is determined by integration of the effect of these forces over the length of the specimen, with a factor added to include the effect produced by the mass of the piston. Finally, an expression for the stress is obtained by dividing the total bending moment by the proper section modulus. As mentioned above, this general expression for the maximum bending stress is derived in the convenient form of:

$$S = KA_m'f^2$$

where

S	stress, psi
K	determinable constant
A_m'	maximum amplitude of vibration at free end of fatigue specimen, inches
f	natural frequency of vibration, cycles per second

The determination of the bending moment will be considered first.

(1) Bending moment. In harmonic motion, the displacement of a point varies in the following manner:

$$X = A_m \sin \omega t$$

where

X	displacement, inches
A_m	maximum amplitude at any point along specimen, inches
ω	angular frequency, cycles per second $(2\pi f)$
t	time, seconds

Thus, the acceleration of a point which is the second derivative of its displacement with respect to time is given by:

$$\ddot{X} = -\omega^2 A_m \sin \omega t$$

or

$$|\ddot{X}| = \omega^2 A_m$$

where $\ddot{X}$ is acceleration.

Force, which is mass times acceleration, becomes in this case:

$$F = m\ddot{X} = m\omega^2 A_m$$

where

F	force, pounds
m	mass, slugs

and, therefore, the bending moment is:

$$M = l m\omega^2 A_m$$

where

M	bending moment, inch-pounds
l	moment arm of force, that is, distance from vibrating point to fixed end, inches

Thus far, only one point on the specimen has been considered to illustrate the factors upon which the bending moment depends.

The total bending moment must be obtained by integrating over all points along the length of the specimen. The maximum displacement or amplitude at each point along the length of the specimen depends upon its distance to the fixed end. This dependence can be stated mathematically by the following series:

$$A_m = A_m' \left[s\left(\frac{l}{L}\right) + r\left(\frac{l}{L}\right)^2 + t\left(\frac{l}{L}\right)^3 + u\left(\frac{l}{L}\right)^4\right]$$

where

A_m	maximum amplitude at point along specimen, inches
A_m'	maximum amplitude at free end of specimen, inches
s,r,t,u	constants which can be determined from deflection curve
l	distance from point along specimen to fixed end, inches
L	length of specimen, that is, from free end to fixed end, inches

The differential equation for the bending moment is

$$dM = l m \omega^2 dA_m$$

and

$$dA_m = A_m'\left(\frac{s}{L} + \frac{2rl}{L^2} + \frac{3tl^2}{L^3} + \frac{4ul^3}{L^4}\right) dl$$

Inserting the expression for dA_m in the equation for dM yields

$$M = m\omega^2 A_m' \int_0^L \left[s\left(\frac{l}{L}\right) + 2r\left(\frac{l}{L}\right)^2 + 3t\left(\frac{l}{L}\right)^3 + 4u\left(\frac{l}{L}\right)^4\right] dl$$

which, upon integrating, becomes

$$M = m\omega^2 A_m' L\left(\frac{s}{2} + \frac{2r}{3} + \frac{3t}{4} + \frac{4u}{5}\right)$$

But the mass of the specimen is

$$m = \frac{W}{g} = \frac{\rho V}{g} = \frac{\rho L d h}{g}$$

where

m	mass of specimen, slugs
W	weight of specimen, pounds
ρ	density of material, pounds per cubic inch
V	volume of specimen, cubic inches
g	acceleration due to gravity, inches per second per second
L	length of specimen, inches
d	width of specimen, inches
h	thickness of specimen, inches

Substituting the proper expression for the mass of the specimen in the bending-moment equation yields

$$M = \frac{\rho dhL^2\omega^2 A_m'}{g}\left(\frac{s}{2} + \frac{2r}{3} + \frac{3t}{4} + \frac{4u}{5}\right)$$

This equation gives the contribution to the bending moment made by the specimen itself.

The bending moment produced by the mass of the piston attached to the end of the fatigue specimen must now be considered. This portion of the bending moment is simply as follows:

$$M = \frac{W'}{g}\,\omega^2 \times yA_m' \times pL$$

where

W' weight of piston, pounds

ω angular frequency of vibration, cycles per second

yA_m' amplitude of vibration at center of mass of piston, inches

pL distance from center of mass of piston to fixed end, inches

The fractions y and p are both easily measured.

Combining the two contributions to the bending moment:

$$\sum M = \omega^2 A_m'\left[\frac{\rho dhL^2}{g}\left(\frac{s}{2} + \frac{2r}{3} + \frac{3t}{4} + \frac{4u}{5}\right) + \frac{W'ypL}{g}\right]$$

(2) Section modulus. The section modulus is simply

$$Z = \frac{I}{c}$$

where

Z section modulus, inches^3

I moment of inertia, inches^4

c distance from outermost fiber to neutral axis, inches

For the present case

$$I = \frac{1}{12} dh^3$$

where

d width of specimen, inches

h thickness of specimen, inches

and

$$c = \frac{h}{2}$$

Thus,

$$Z = \frac{dh^2}{6}$$

(3) Stress. Having obtained expressions for the total bending moment and the section modulus, the stress may now be calculated by dividing

$$S = \frac{\sum M}{Z}$$

$$S = f^2 A_m' \left\{ \frac{24\pi^2 L}{hg} \left[\rho L \left(\frac{s}{2} + \frac{2r}{3} + \frac{3t}{4} + \frac{4u}{5} \right) + \frac{W' y_p}{dh} \right] \right\}$$

This is a complete analytical expression for the calculation of the stress in a vibrating-cantilever specimen. The various constants are evaluated from the deflection curve and the physical dimensions of the test bar to yield a general equation in the convenient form

$$S = Kf^2 A_m'$$

With this equation, the stress may be readily calculated by inserting the measured values of frequency and amplitude for a given specimen design and material.

(4) Deflection curve. As indicated above, knowledge of the deflection curve is necessary in order to evaluate the constants in the stress equation. This curve is merely the displacement of points along the specimen from their normal nonvibrating positions as a function of the distance from the fixed end. Such a curve may readily be obtained by direct measurements of the amplitudes of vibration at various points along the length of the specimen. In determining the deflection curves, care was taken that all deflections were within the elastic range of the material. From such measurements, the following data were obtained for annealed Armco iron:

A_m/A_m'	l/L
1	1
.900	.909
.633	.721
.433	.555
.233	.333

Inserting these values in the equation:

$$\frac{A_m}{A_m'} = \left[s\left(\frac{l}{L}\right) + r\left(\frac{l}{L}\right)^2 + t\left(\frac{l}{L}\right)^3 + u\left(\frac{l}{L}\right)^4\right]$$

four simultaneous equations are obtained which can be solved by the method of determinants to yield the values of the four constants. These values are:

$$s = 1.000$$

$$r = -2.142$$

$$t = 4.428$$

$$u = -2.190$$

(5) Constant K. Having determined the deflection constants (s, r, t, and u), it is now possible to calculate the over-all constant K.

The various quantities involved are as follows:

$L = 4.502$ in.	$r = -2.142$
$h = 0.300$ in.	$t = 4.428$
$d = 0.500$ in.	$u = -2.190$
$\rho = 0.284$ lb/cu in.	$W' = 0.314$ lb
$g = 386$ in./sec^2	$y = 0.900$
$s = 1.000$	$p = 0.909$

Inserting these values in the stress equation given previously, the following result is obtained:

$$S = 23.298 A_m' f^2$$

Stress values determined by the strain-gage method agreed with the equation to within 2 percent.

It should be realized that the value for the constant K in the above equation applies only to annealed Armco iron which was used to obtain the deflection curve. For each new type of material it was necessary to repeat the deflection-curve measurements, determine the set of deflection constants, and calculate the proper value of K. It was found that the same deflection curve was obtained for the SAE 4340 steel in the two heat-treated conditions. The stress equation for the SAE 4340 materials is as follows:

$$S = 30.694 A_m' f^2$$

Estimate of error: The calculation of stress depends upon a number of quantities such as amplitude, frequency, specimen dimensions, and so forth. Since these quantities are measured and therefore subject to an uncertainty, a corresponding error can be introduced in the calculated stress. The possible extent of this error can easily be calculated.

First consider the error in stress due to the uncertainty in the frequency measurement:

$$dS = \left(\frac{\partial S}{\partial f}\right) df$$

or

$$\text{Percentage error} = \frac{dS}{S}\,100 = \frac{\partial S}{\partial f} \times \frac{df}{S} \times 100$$

Since $S = KA_m'f^2$,

$$\frac{\partial S}{\partial f} = 2KA_m'f$$

Thus,

$$\text{Percentage error} = \frac{\left(2KA_m'f\right)\,df}{KA_m'f^2} \times 100 = 2\left(\frac{df}{f}\right)100$$

The uncertainty in frequency df is 1 cycle per second and in the frequency itself is approximately 200 cycles per second.

Therefore,

$$\text{Percentage error} = \frac{2(1)}{200} \times 100 = 1.0 \text{ percent}$$

Similarly, the error in stress due to the uncertainty in other measurements and variations within machining tolerances can be calculated. The results for the vibrating fatigue test are as follows:

Measured quantity	Percentage error in stress
Frequency	1.00
Amplitude	.50
Specimen length	.05
Specimen width	.11
Specimen thickness	.42
Total possible error	2.08

The effect of experimental error will be considered subsequently.

Experimental Results and Analysis

The fatigue data for annealed Armco iron and the SAE 4340 steel heat-treated to two different structures were analyzed by the same statistical methods outlined in part I.

The statistics of the fracture curve and endurance limit for Armco iron are given in the two following tables and plotted in figures 28 and 29. It will be noted that there is considerable dispersion in the results and that the mean S-N curve has a sigmoid form.

Stress (psi)	Mean of logs of life values, $\overline{\log N}$	Mean life, $\overline{N}$ (cycles)	Unbiased standard deviation, σ	Standard estimate of error, $\sigma_{\overline{\log N}}$	Relative standard deviation, V (percent)
41,750	5.76912	5.8765×10^5	0.1649	0.0368	2.85
38,770	6.27350	1.8772×10^6	.1519	.0339	2.42
35,790	6.41049	2.5733	.2512	.0576	3.92
32,800	6.85432	7.1502	.2634	.0639	3.84

Stress (psi)	Failures (percent) (1)
32,800	90
29,820	50
26,840	10

[1]Mean endurance limit, 29,820 psi;
σ endurance limit, 2328 psi.

The statistics of the fatigue properties of quenched and spheroidized SAE 4340 are presented in the next two tables and plotted in figure 30.

Stress (psi)	Mean of logs of life values, $\overline{\log N}$	Mean life, $\overline{N}$ (cycles)	Unbiased standard deviation, σ	Standard estimate of error, $\sigma_{\overline{\log N}}$	Relative standard deviation, V (percent)
81,430	5.09283	1.238×10^5	0.1235	0.0269	2.42
74,650	5.51555	3.277	.2353	.0526	4.26
70,570	5.57179	3.730	.1922	.0430	3.45
65,150	5.68845	4.880	.2941	.0735	5.17
62,430	6.03732	1.089×10^6	.3534	.1180	5.85

Stress (psi)	Failures (percent) (1)
65,150	80
62,430	50
59,720	10

[1]Mean endurance limit, 62,430 psi; σ endurance limit, 2490 psi.

Similarly, the results for quenched and tempered SAE 4340 are given below and in figure 31.

Stress (psi)	Mean of logs of life values, $\overline{\log N}$	Mean life, $\overline{N}$ (cycles)	Unbiased standard deviation, σ	Standard estimate of error, $\sigma_{\overline{\log N}}$	Relative standard deviation, V (percent)
111,270	5.11527	1.3040×10^5	0.1704	0.0381	3.33
104,490	5.39605	2.4892	.2630	.0588	4.87
100,420	5.42988	2.6908	.2208	.0637	4.06
97,700	5.60586	4.0352	.2688	.0808	4.79

Stress (psi)	Failures (percent) (1)
100,420	60
97,700	52.4
94,990	30

[1]Mean endurance limit, 97,600 psi; σ endurance limit, 7070 psi.

Statistics of fatigue-fracture curve.- One of the purposes of this investigation is to determine the influence of metallurgical factors on the statistical variation in the fatigue life. This can be accomplished by comparing the fracture-curve statistics for the various materials with different compositions, heat treatments, microstructures, and inclusion ratings. However, such comparisons require a choice as to the basis on which they will be made. Obviously, the analysis of variability must be performed under equivalent conditions. The endurance limits of the various materials are different and, moreover, there is a possibility that the dispersion in fatigue life for a given material depends upon the stress level. Thus, the latter possibility must be examined in some detail before the basis of analysis can be chosen and the comparisons made.

This problem of the dependence of dispersion in the fracture range on stress level was considered briefly in part I of this investigation, but a more complete analysis is needed, particularly to include additional results for other materials and methods of test. In part I, the Hartley method was used, but here the F test will be applied since it is simpler to use and is more informative for the present purpose. The F test (reference 16) is a means of comparing the spreads or variabilities of two sets of measurements (in this case, the variability in fatigue life at high stress with that at low stress for a given material) to determine whether these variabilities differ significantly at some chosen level of confidence. The procedure is as follows:

(1) Calculate σ, the unbiased standard deviation, for the fatigue life at each stress level in the fracture range.

(2) Square each of the σ values to get the variance.

(3) Divide σ^2 for the fatigue life at one stress level by σ^2 for another level (placing the larger value in the numerator) to obtain the variance ratio.

(4) Compare this result with the proper value in the F test tables for the degrees of freedom involved and the chosen level of confidence. The degrees of freedom are one less than the number of specimens used. The level of confidence chosen in this work was the 5-percent level so that there are 19 chances out of 20 that the conclusion drawn is correct.

(5) If the calculated variance ratio is greater than the value given in the table, it can be concluded that the difference in fatigue-life variability at the two stress levels is significant at the 5-percent level.

This procedure was applied to all available fatigue data in the fracture range in which a sufficient number of specimens was used to permit statistical analysis. The results of this analysis are summarized in the following table. (The abbreviations Q&S and Q&T indicate quenched and spheroidized specimens and quenched and tempered specimens, respectively.)

Material	Source	Type of test	Result of F test at 5-percent level
Armco iron	Present investigation	Flexure	Significant
SAE 4340 (Q&S)	-----------do---------	-----do----	Do.
SAE 4340 (Q&T)	-----------do---------	-----do----	Do.
SAE 1050	-----------do---------	R. R. Moore	Do.
SAE 4340 (Q&T)	Reference 8	-----do----	Do.
75S-T	Reference 4	-----do----	Do.
SAE 1045	-----------do---------	-----do----	Do.
SAE 4340	-----------do---------	-----do----	Do.
Armco iron	Reference 6	Torsion	Not significant
Copper	-----------do---------	-----do----	Do.
Aluminum	-----------do---------	-----do----	Do.

The analysis discloses the fact that in R. R. Moore and vibrating-cantilever fatigue tests the variability in fatigue life does depend upon the stress level in the fracture range with greater dispersion occurring at the lower stress. In torsional fatigue tests, however, the dispersion in fatigue life is independent of the stress level. A theoretical interpretation of this interesting and important behavior will be presented in the "Discussion" section.

It is evident from the behavior shown above that comparison of fatigue-life variabilities for two materials (tested in rotation or flexure) must be made at equivalent stress levels. Differences in endurance limits between materials can be eliminated by calculating fracture stress levels as a percentage of the mean endurance limits and comparisons of variability can be made at each of the various percentage stress levels. It was on this basis that the influence of metallurgical factors on the dispersion in fatigue life was analyzed.

In figure 32 is shown a summary plot of the statistics of the fracture curves for most of the available results of R. R. Moore and vibrating-cantilever tests. It is not possible to draw conclusions from these data with a high degree of confidence without applying mathematical tests (such as the F test), but preliminary visual examination indicates a few points of interest, for example:

(1) The SAE 1050 values seem to fall on two curves suggesting that the specimens are derived from two populations. This, however, is not true. The Hartley test was applied to these data in part I and it was found that the values are homogeneous and therefore derived from one parent population.

(2) There is a trend (proven above by rigorous methods) toward higher dispersion in fatigue life with decrease in stress toward the endurance limit. This trend exists in the data obtained from both the R. R. Moore machines and the pneumatic vibrating-cantilever machines which have different criteria for failure (complete fracture and crack initiation, respectively).

(3) The dispersion in fatigue life does not depend upon composition alone. Values for SAE 4340 lie toward the top of the plot while those for another lot of SAE 4340 (data from reference 8) lie at the lower portion of the figure. Factors other than composition must play an important role in this behavior.

The F test was applied to determine whether the differences in variance for the various materials were significant at the 5-percent level of confidence. Comparison was made at equivalent stress levels (i.e., at equal percentages of the mean endurance limit). The results of the analysis are summarized as follows:

Materials compared (1)	Result of F test at 5-percent level
Armco iron (reference 6) - Copper (reference 6)	Significant
Armco iron (reference 6) - Aluminum (reference 6)	Do.
Aluminum (reference 6) - Copper (reference 6)	Not significant
SAE 4340 (Q&S) - SAE 4340 (Q&T)	Do.
SAE 4340 (Q&S) - SAE 4340 (reference 8)	Significant
SAE 4340 (Q&S) - SAE 1050	Not significant
SAE 4340 (reference 4) - SAE 4340 (Q&S)	Do.
SAE 4340 (reference 4) - SAE 4340 (reference 8)	Significant
SAE 4340 (Q&T) - SAE 4340 (reference 8)	Do.
SAE 4340 (Q&T) - SAE 1050	Not significant
SAE 4340 (Q&T) - Armco iron	Do.
Armco iron - SAE 4340 (Q&S)	Do.
Armco iron - SAE 4340 (reference 8)	Significant
Armco iron - SAE 1050	Not significant

[1]Material listed first had larger numerical variance.

As an aid in the interpretation of these results, the available metallurgical data for the materials are summarized in the following table:

Material	Heat treatment	Microstructure	Rockwell hardness	Tensile strength (psi)	SAE inclusion rating
Armco iron	Annealed	Ferrite	B-53	42,200	$2^{vd} - 1.1^{9vd}$ - D
SAE 1050	Normalized and tempered	Ferrite with partially tempered pearlite	B-81	91,400	$3^{d} - 1.4^{300d}$ - C
SAE 4340 (One heat)	Quenched and spheroidized	Ferrite and fine spheroidite	B-98	103,482	$4^{vd} - 1.7^{28vd}$ - B
	Quenched and tempered	Ferrite and tempered martensite	C-38	167,790	$4^{vd} - 1.7^{28vd}$ - B
SAE 4340 (Reference 8)	Quenched and tempered	Ferrite and tempered martesite	C-25	125,000	0 - 0 - A
SAE 4340 (Reference 4)	Quenched and tempered	Ferrite and tempered martensite	C-35	163,700	Unknown, but probably higher than 0 - 0 - A

The conclusions which can be drawn from the results given in these tables and their interpretation will be considered in the "Discussion" section.

Statistics of endurance limit.- The only data available on the statistical variation in the endurance limit are those from Ransom's work (reference 8) and the present investigation. These results are given in the following table:

Material	Mean endurance limit, $\bar{S}$ (psi)	Standard deviation, σ (psi)	Relative standard deviation, V (percent)
[a]SAE 1050	40,800	1028	2.52
Armco iron[a]	29,820	2328	7.80
SAE 4340 (Q&S)[a]	62,430	2490	4.00
SAE 4340 (Q&T)[a]	97,600	7070	7.32
SAE 4340 (Ransom)			
High RAT heat:			
Longitudinal[b]	61,750	1980	3.20
Transverse[b]	52,400	1580	3.02
Transverse[c]	52,480	1540	2.94
Transverse[d]	51,600	4720	9.15
Low RAT heat:			
Longitudinal[b]	67,040	1700	2.50
Transverse[b]	46,270	2900	6.25
Transverse[c]	46,280	3140	6.80
Transverse[d]	46,820	5300	11.3

[a]Results of present investigation based on probit method.

[b]Results obtained by "staircase method" as explained by Ransom (reference 8).

[c]Results obtained from probit analysis of data for same specimens used in staircase method.

[d]Results obtained from probit analysis of approximately twice as many specimens as used in staircase method (including those used in latter treatment).

Analysis of a given limited set of data by the "staircase" and probit methods yield essentially the same results, as can be seen in the transverse tests reported by Ransom. However, when the number of tests was roughly doubled, the probit method gave much larger estimates of the variability (see the transverse values). This situation makes it somewhat difficult to compare Ransom's results with those obtained in the present work. The latter is concerned only with the longitudinal direction and results were analyzed by the probit method; Ransom, on the other hand, did not get values for the longitudinal direction by the probit method, which makes comparison rather uncertain. Thus, for the most part, the comparisons in endurance-limit statistics will have to be restricted to the individual investigations.

Tests of significance must be applied to these data to facilitate the determination of the influence of metallurgical factors on the variability in endurance limit. Usually the F test is applied to test for a significant difference in variance. In the present case, however, the F test is not useful, particularly because the proper number of degrees of freedom is highly uncertain. For example, suppose that 100 specimens were tested at each of four stress levels in the endurance range. The percentage of failures at each level would be noted and analysis of the response by the probit method would yield the mean endurance limit and its standard deviation. In using the F test to compare this standard deviation with that obtained for another material from a similar set of experiments, a choice must be made of the proper number of degrees of freedom. Ordinarily the degrees of freedom used in the F test is one less than the number of specimens, but in the present case 400 - 1 would be too large because the value of σ depends upon the response of four groups. On the other hand, taking one less than the number of stress levels used (4 - 1) would be too small, since this value does not take into account the large number of specimens used at each stress level. Because of this difficulty the F test cannot be applied in the present case.

The formal treatment of the probit method does not include a method of testing for the significance of differences in variability of response; it does, however, give the variance of the slope of the probit line. (It should be recalled that the reciprocal of this slope is the standard deviation in the endurance limit.) Thus, a test of significance can be applied to the slopes of the probit lines and, if these differ significantly, then it can be said that the standard deviations (reciprocals of the slopes) also differ significantly.

The method used for this purpose was the t test, assuming normal distribution and large samples (reference 16). This solution will be illustrated by the following example:

The slope of the probit straight line and its variance for Armco iron are 4.28×10^{-4} and 3.21×10^{-10}, respectively. The corresponding values for SAE 4340 (quenched and tempered) are 1.41×10^{-4} and 14.83×10^{-10}. The expression for t is given by:

$$t = \frac{b_1 - b_2}{\sqrt{V_{b_1} + V_{b_2}}}$$

where

b slope

V_b variance of b

Substituting the values for the two materials yields:

$$t = \frac{4.28 \times 10^{-4} - 1.41 \times 10^{-4}}{\sqrt{3.21 \times 10^{-10} + 14.83 \times 10^{-10}}} = 6.7$$

The value of t for the 95-percent level of confidence for 2 degrees of freedom (i.e., $3 + 3 - 2 - 2 = 2$) is 4.30. Thus, since 6.7 is greater than 4.30, it can be concluded with 95-percent confidence that the difference in variability of the endurance limits is significant.

Similar calculations were made for the other materials, the results of which are summarized as follows:

Materials compared (1)	Results of t test at 5-percent level
Armco iron - SAE 1050	Significant
SAE 4340 (Q&S) - Armco iron	Not significant
SAE 4340 (Q&S) - SAE 1050	Significant
SAE 4340 (Q&T) - Armco iron	Do.
SAE 4340 (Q&T) - SAE 4340 (Q&S)	Do.
SAE 4340 (Q&T) - SAE 4340 (high RAT)	Do.
SAE 4340 (Q&T) - SAE 4340 (low RAT)	Do.
SAE 4340 (Q&T) - SAE 1050	Do.
SAE 4340 (high RAT) - SAE 4340 (low RAT)	Not significant

[1]Materials with larger standard deviation in endurance limit are listed first. Data for high and low RAT SAE 4340 taken from reference 8.

These results will be considered in the "Discussion" section.

S-N diagram.- The mean fatigue properties of the various materials tested in this investigation, as well as Ransom's results, have been plotted in complete S-N diagrams by the method proposed in part I. It is seen in figures 33 to 36 that the method is not so successful as when applied to Wallgren's data (fig. 18). Straight lines are obtained in these plots, but these lines do not pass through the tensile-strength point at 1/4 cycle. This matter will be discussed in a later section.

Understressing study.- As an extension of the investigation of the statistical behavior of the fatigue of metals, a start was made on the study of the so-called understressing effect. This study was undertaken because it did not require the use of any additional specimens and provided the opportunity to obtain additional information on the statistical nature of fatigue properties with only relatively little extra effort.

Those specimens which did not fail in fatigue test at stresses in the endurance range were, upon successful completion of the specified number of cycles without failure (2.5×10^7 cycles), retested at a stress in the fracture range for which the life of the virgin material was known. In the case of the two lots of SAE 4340, the retest at high stress was performed immediately after the endurance run without removal of the specimen from the machine. The idea of conducting such tests was conceived long after the fatigue testing of Armco iron had been completed so that in this case there was an interval of approximately 4 months between the time a specimen had run out in endurance test and the time it was retested at a stress which produced failure. Thus, the Armco specimens were subjected to an intermediate rest period at room temperature for 4 months.

The number of such tests conducted was restricted to the number of specimens which did not fail in the endurance tests, but this included in most cases a sufficient number to indicate the trends.

In conducting these tests, a record was kept of the stress level at which the specimen had been tested in the endurance range and the subsequent fatigue lives at the higher stress were grouped on this basis. The following experimental results were obtained:

For annealed Armco iron (mean endurance limit, 29,820 psi):

Condition	Mean fatigue life, $\bar{N}$, at 41,750 psi (cycles)	Unbiased standard deviation, σ	Number of specimens
Virgin material	5.876×10^5	0.1649	20
Understressed at 32,800 psi	9.700	.1207	2
Understressed at 29,820 psi	8.493	.2830	5
Understressed at 26,840 psi	8.481	.2420	12

For quenched and spheroidized SAE 4340 (mean endurance limit, 62,430 psi):

Condition	Mean fatigue life, $\bar{N}$, at 74,650 psi (cycles)	Unbiased standard deviation, σ	Number of specimens
Virgin material	3.277×10^5	0.2353	20
Understressed at 65,150 psi	8.310	.3087	4
Understressed at 62,430 psi	3.812	.1809	10
Understressed at 69,720 psi	3.272	.2667	18

For quenched and tempered SAE 4340 (mean endurance limit, 97,600 psi):

Condition	Mean fatigue life, $\bar{N}$, at 111,270 psi (cycles)	Unbiased standard deviation, σ	Number of specimens
Virgin material	1.3040×10^5	0.1704	20
Understressed at 100,420 psi	2.7370	.2780	8
Understressed at 97,700 psi	2.5177	.2211	7
Understressed at 94,990 psi	2.1560	.2318	14

These data show that, in most cases, understressed specimens have longer fatigue life than virgin specimens at a given stress in the fracture range.

The effect can be established by application of the t test (reference 33). This test is a method of comparing two means and determines whether the mean of one set of measurements is significantly different from the mean of another set. The procedure is to calculate a value of t for the data and compare this value with that obtained from the table of t values for the proper number of degrees of freedom and the chosen level of confidence. If the calculated value exceeds the tabulated value for the 5-percent level, it can be concluded with that degree of assurance that the two means are different and that this difference could not be due to chance alone.

The t test involves the following relationships:

$$\bar{X} = \frac{\sum(x)}{n_1}$$

$$\bar{X}' = \frac{\sum(x')}{n_2}$$

where there are n_1 observations of x in one set and n_2 observations of x' in the other set.

$$\sigma^2 = \frac{\sum(x'^2) - \frac{\left[\sum(x')\right]^2}{n_2} + \sum(x^2) - \frac{\left[\sum(x)\right]^2}{n_1}}{n_1 + n_2 - 2}$$

$$t = \frac{\bar{X} - \bar{X}'}{\sigma} \times \frac{n_1 n_2}{n_1 + n_2}$$

The results of t test analysis of the understressing data are summarized in the following table:

Material	Understress (psi)	Result of comparison with mean virgin life at 5-percent level
Annealed Armco iron	32,800	Not significant (significant at 10-percent level)
	29,320	Not significant (significant at 15-percent level)
	26,840	Significent
SAE 4340 (Q&S)	65,150	Significant
	62,430	Not significant
	59,720	Do.
SAE 4340 (Q&T)	100,420	Significant
	97,700	Do.
	94,990	Do.

It is seen from the preceding tables that understressing may increase the subsequent fatigue life at an elevated stress relative to the virgin life and that in many cases this increase is significant and not due to chance alone. An experimental difficulty arises in this study because of the limited number of run-out specimens available. It would be desirable to have 20 run-out specimens from each of the three stress levels in the endurance range in order to obtain a good estimate of the behavior by methods of statistical analysis. This requirement, however, would be extremely difficult to meet. For example, in order to obtain 20 run-out specimens of Armco iron understressed at 32,800 psi, it would be necessary to test 200 specimens, since only 10 percent of the bars tested at this stress level do not fail.

The significance of these understressing results and their interpretation will be discussed at length in a later section.

Discussion

Metals in reality are highly imperfect. Single crystals of pure metals contain lineage and mosaic faults of growth as well as dislocations and vacancies. While the faults of growth can sometimes be decreased or eliminated by subsequent treatment, such as recrystallization, a theoretically perfect crystal is not obtainable since real metals contain an equilibrium number of vacancies. The addition of an alloying element in solid solution introduces the possibility of further imperfection in compositional segregation. In the case of crystal aggregates, grain boundaries are regions of misfit, the degree of which depends upon the mutual orientations of the crystals; in aggregates which contain a second phase, there are also imperfections in microstructure. The usual commercial material contains any or all of these plus inclusions, an additional and serious type of imperfection.

For a given class of material, the types of imperfections and their relative magnitudes can differ widely and, further, the importance of a given type may differ among the various classes of materials. The problem becomes one of recognizing the important types of imperfections, describing each of them as to severity and frequency distribution, and considering the effects of this set of conditions. The existence of a universe of imperfections in a metal has been generally accepted, but a demonstration (both experimental and theoretical) of the consequences of this universe has been lacking. The statistical nature of fatigue properties dramatically demonstrates the behavior.

Imperfections play an important role in the mechanism of fatigue failure. Unlike static loading, in which stress concentrations are reduced by plastic flow, the successive fatigue cycles work-harden the material in the neighborhood of imperfections so that ultimately failure may occur. Thus, owing to the presence (or scarcity) of imperfections, it is easy to visualize a distribution in quality of specimens derived from one source ranging from poor, through average, to superior fatigue strength. This variation in quality is reflected in the dispersion in fatigue life and the statistical nature of the endurance limit. The latter case illustrates the behavior quite nicely. As the level of test stress in the endurance range is progressively decreased, that is, as the quality requirement for nonfailure is lowered, an increasing number of specimens do not fail.

The details of this behavior have not been well-understood and it was toward this subject that much of this investigation was directed. It will be shown that, on the basis of the statistical nature of fatigue properties and the distribution of imperfections in metals, many new phenomena can be detected and explained.

Dependence of statistical variation in fatigue life on stress level.- It has been shown in a previous section that the statistical variation in fatigue life in rotating and vibratory tests depends upon the stress level with a definite increase in dispersion with decrease in stress within the fracture range. In torsional tests, however, the dispersion in fatigue life does not depend upon the stress level to any significant degree. These facts are newly established and are in need of theoretical interpretation.

Consider the frequency distribution of effective imperfections in a metal. This distribution can be assumed to have the form of the curve shown in figure 37, with a large number of very small imperfections and a decrease in population with increase in size. These imperfections can be any of those outlined above, but inclusions are probably one of the more serious and important types. It has been observed metallographically that fatigue cracks in steels most frequently initiate at an inclusion and, as will be shown subsequently, inclusions are probably the most important single factor in determining the extent of statistical behavior. Thus, in the present argument, the terms "imperfections" and "inclusions" will be used interchangeably.

For a given stress level in the fatigue-fracture range, there is a critical minimum size (or severity) of inclusion which can be effective in promoting fatigue failure (of course, those larger than this critical minimum size can also be effective). For a high stress, this critical size of inclusion is relatively small, while at lower stress the size of the inclusions which can be effective is larger. Thus, as seen in figure 37, the total number of effective inclusions available at high stress is greater than in the case of low stress.

In a rotating or vibratory fatigue specimen, failure usually initiates in the small volume of metal which is subjected to maximum stress. Indeed, the occurrence of failure at a point slightly removed from the point of maximum stress is attributable to the presence of a very large inclusion or inhomogeneity. Thus, fatigue failure and its statistical variation depend upon the probability of occurrence of effective inclusions in this small region of the test specimen, at or near maximum stress.

At a high applied stress, a large number of effective inclusions are available and the probability of occurrence of effective inclusions in the region of maximum stress is high. On the other hand, at low applied stress the total number of effective inclusions is less and the probability of encountering active inclusions in the critical region is lower; in this case some specimens have several effective inclusions while others have few or perhaps even none.

This state of affairs gives rise to different dispersions in fatigue life at the two stress levels. As shown schematically in figure 38, there is a distribution in fatigue life for specimens with many inclusions and another distribution with a higher mean life for specimens with few inclusions. At a high applied stress, a distribution in fatigue life is obtained which is related to that for many inclusions. At a low stress, however, the distribution in fatigue life obtained is related to that associated with both large and small quantities of inclusions, as shown in figure 38, because at this stress level some specimens have a number of effective inclusions while others have few or none.

The fact that the dispersion in fatigue life obtained in torsion tests is independent of the stress level can be understood by considering the nature of the test. The torsion test, unlike the rotating- or vibrating-cantilever test, produces a uniform stress along the entire gage length of the specimen. Thus, in this test there is not a small critical region at maximum nominal stress and the probability of occurrence of an inclusion of effective size in the test region is high at all stresses; active inclusions are always certain to be present. This argument is supported by the experimental observation that only one crack forms in the R. R. Moore and vibrating-cantilever test specimens while a number of cracks are produced in specimens subjected to tests in which the stress is uniform along the gage length.

Thus, in torsional fatigue, the distribution in fatigue life at all stresses corresponds to the "many inclusions" situation which exists in the case of high stresses in rotating and vibrating tests. Therefore, this hypothesis predicts that the statistical variation in fatigue life in torsional tests at all stresses in the fracture range should be the same as that obtained with the given material tested in rotation or flexure at high stresses. This prediction can be checked by comparing the standard deviations in the fatigue life of annealed Armco iron obtained in flexural tests at two high stresses ($\sigma = 0.16$ and 0.15) with the average of those calculated from Ravilly's torsional tests at all stress levels ($\sigma = 0.15$). This agreement is even better than expected.

This problem of variability, the effect of imperfections, and the influence of having a limited volume of the specimens at maximum nominal stress have a direct bearing on the size effect in fatigue which will be discussed later in this section.

While the above argument has been based on a material with a relatively high inclusion rating, the same considerations can be applied to a clean steel. The distribution curve for imperfections in the latter material lies below that for the dirty steel, as shown schematically in figure 39. By considering the total number of imperfections available at high and low stresses, and the relative probabilities of encountering effective imperfections in the critical-stress volume, the dispersion in fatigue life and its change with stress level can be foreseen. In figure 40, it is indicated that, in the case of a clean steel, the dispersion in fatigue life may increase with decrease in stress level, but the effect is not so marked as in the case of a dirty steel. Further, it seems likely that, at very high stresses, the dispersion in fatigue life of clean and dirty materials should not differ very greatly.

In figure 32, it is seen that the standard deviation in fatigue life of the clean heat of SAE 4340 (Ransom's data from reference 8) increases somewhat with decrease in stress level and the effect is less marked than in the case of a dirty heat. In addition, it appears that the differences in variability between clean and dirty materials tend to decrease at the highest stress levels.

Influence of metallurgical factors on statistical behavior.- Study of the information given in the second and third tables of the section "Statistics of fatigue-fracture curve" discloses the fact that inclusions are probably the most important single factor in determining the extent of the statistical variation in fatigue life. This conclusion is established from the following:

(1) It is shown that two SAE 4340 heats of identical chemical composition but with different inclusion ratings have correspondingly different dispersions in fatigue life; large variability is associated with high inclusion rating.

(2) Differences in microstructure (e.g., SAE 4340 quenched and tempered against same heat quenched and spheroidized) for a given material and inclusion rating do not produce significant differences in fatigue-life variability.

(3) Differences in composition alone (e.g., Armco iron against SAE 4340) do not produce differences in fatigue-life dispersion provided the inclusion ratings are similar. If, however, the inclusion ratings of the materials differ (e.g., Armco iron against SAE 4340 (reference 8)), the statistical variation in fatigue life differs correspondingly.

While it was believed at the start of this investigation that inclusions would be one of several factors which could influence the statistical behavior of fatigue properties, it was not suspected that this factor would be by far the most important of the variables investigated. Alloy segregates and other compositional factors which were thought to have some importance have little if any effect. Further, it is somewhat surprising to discover that the variability in life is not significantly affected by microstructure, other things being equal. It might well be expected that an acicular structure would yield greater variability than that obtained with a spheroidal structure or a single phase; the results, however, indicate that this is not true or at least the effect is of minor importance, since these factors are completely overshadowed by the inclusion effect.

Another result which was wholly unanticipated is the absence of difference in fatigue-life variability obtained at different hardness levels for a given inclusion rating. Experiments on a heat of SAE 4340 with a high inclusion rating and heat-treated to Rockwell hardness C-38 and B-98 yielded essentially the same variabilities along the fracture curve. It would not have been unreasonable to predict that greater variability in fatigue life would be obtained at the higher hardness level, but the present experiments did not disclose such a difference.

Comparison of results calculated from Ravilly's fatigue tests on Armco iron, copper, and aluminum shows that the nonferrous materials have significantly lower dispersion in fatigue life than that obtained with Armco iron. While the relative inclusion ratings of these materials were not reported, it may be assumed that the nonferrous metals have lower inclusion contents than the iron.

An observation of some importance is that, on the whole, the variabilities in fatigue life and endurance limit show the same trends. That is, if the dispersion in fatigue life of one material is significantly greater than that for another, then the variability in the endurance limits differs significantly in the same manner. From an experimental viewpoint this situation, if invariably true, would be quite useful since the relative variabilities of materials could be determined by fracture tests alone without performing the more laborious experiments in the endurance range. However, it was found in at least one case that two materials with essentially the same fracture-curve statistics have different endurance-limit statistics. Thus, the use of experimental short cuts would be subject to uncertainties.

The results of tests for significance of differences in the variability in endurance limits were summarized by the second table included in the section "Statistics of endurance limit." Here again it is found that the predominant factor in producing variability is the inclusion content and many of the same conclusions may be drawn as in the discussion of the fatigue life. In the present case, however, unlike the results for fatigue-life dispersion, the standard deviations for SAE 4340 quenched and tempered and the same heat quenched and spheroidized differ significantly. As noted just previously, it might be expected that with a given inclusion rating an acicular structure would yield greater dispersion than a spheroidal structure since the latter is more uniform; further, greater variability might be predicted for the higher hardness level, since as the hardness is increased the effectiveness of the given inclusions would increase. Such differences were not disclosed in the fatigue-life data, but they do exist in the endurance-limit statistics as can be seen by comparing SAE 4340 (quenched and tempered) with SAE 4340 (quenched and spheroidized). Apparently the criterion of failure or nonfailure in the endurance range is more sensitive to such metallurgical differences than is the number of cycles to failure in the fracture range.

There is one rather disturbing discrepancy in the data for the statistics of the endurance limits and that is that the value of standard deviation for SAE 1050 is significantly less than the values for Armco iron or SAE 4340 (quenched and spheroidized). The three materials have fairly high inclusion ratings, Rockwell hardnesses of B-81, B-53, and B-98, respectively, and do not differ significantly with respect to variability in fatigue life. Thus, the SAE 1050 does not differ greatly from the other materials in these important respects and yet has a significantly smaller standard deviation in endurance limit. No explanation for this discrepancy is offered.

Influence of experimental error.- The statistical variations in fatigue life are greater than could be produced by the experimental errors. The maximum possible experimental error in nominal stress is on the order of 1 or 2 percent and, since the S-N curve descends rather steeply (i.e., a given change in stress does not produce so large a change in the logarithm of the number of cycles to failure), this error in stress produces a smaller error in the logarithm of the number of cycles to failure. Thus, the error in measurement of fatigue life is less than 1 percent while the relative standard deviations in fatigue life V obtained from experiment are on the order of 3 or 4 percent. It may safely be assumed, therefore, that the dispersion in fatigue life is inherent in the material and is not simply a result of experimental errors.

Further, the increase in dispersion in fatigue life with decrease in stress in the fracture range is not due to experimental error. It might be argued that an experimental error of say 1 percent would produce an increase in σ with decrease in stress in the fracture range. The mean life increases with decrease in stress in the fracture range and, therefore, it might be said that a 1-percent error in the fatigue life would yield larger dispersions at the lower stresses. If this were true, there would be a similitude in the dispersion along the fracture curve; that is, the relative standard deviations (V values) would be constant through the range. However, as shown in the tables of data for the statistics of the fracture curves, V is not constant but increases with decrease in stress in the fracture range. Therefore, the dependence of σ on stress level is real and not based on experimental error.

The statistical behavior of the endurance limit is also inherent in the material, although experimental error can have an effect. Errors in the nominal stress modify the response of specimens tested in the endurance range from failure to nonfailure and vice versa. In this case too, however, the dispersion obtained is larger than that which could be produced by experimental error alone. The relative standard deviations in endurance limits are from 3 to 7 percent, while the error in stress is on the order of 1 to 2 percent.

The situation with respect to experimental error is even better than the above discussion indicates. A maximum experimental error of ±1 percent corresponds to a $\pm 3\sigma$ range. That is, the lowest experimental deviation is -1 percent, the highest is 1 percent, and the mean error due to experiment is 0; the -1-percent error corresponds to the -3σ level, while the 1-percent error corresponds to the 3σ level, and so forth. Thus, comparison of an experimental error of ±1 percent should be made with the percentage variability related to the $\pm 3\sigma$ range obtained in the data. For example, with a relative standard deviation in fatigue life $\left(V = \dfrac{\sigma}{\log N} \times 100\right)$ of 4 percent, the percentage variability corresponding to the $\pm 3\sigma$ range is ±12 percent. This value is to be compared with the experimental error of ±1 percent. Obviously, experimental error cannot account for the statistical nature of fatigue properties.

In comparing the statistics of fatigue properties of materials, use has been made of data obtained from the R. R. Moore and pneumatic vibratory tests. These two tests utilize different specimen sizes and types and in addition have different experimental errors, being ±1 and ±2 percent, respectively. These differences, if large or important, would invalidate the comparison of data obtained from the two types of test. The differences in specimens, however, are not important. Both types have a limited volume at maximum nominal stress and the difference in this critical test volume is not large. In addition, the difference in experimental error is not great enough to account for the differences in variability of materials tested in the two types of machines. For example, comparison has been made of the statistical variation in the fatigue life of a clean heat of SAE 4340 obtained by Ransom with the R. R. Moore test with that of a dirty heat of the same steel determined in the vibratory test. If the data are corrected for the experimental error in each case, it is found that the difference in dispersion is still significant. Thus, differences in experimental error have not affected the validity of the conclusion drawn from comparison of sets of data obtained from different types of machines.

Theories of failure and statistics.- The role of imperfections in fatigue and the manner in which these imperfections can influence the statistical behavior have already been discussed in some detail. There have been no attempts to explain the statistical nature of fatigue properties on a quantitative theoretical basis. It is felt that the present work has contributed to the fundamental knowledge of fatigue and imposed an additional requirement on the basic theory. Obviously a satisfactory theory of fatigue must explain the mechanism by which failure initiates and, further, in view of recent developments, it must take into consideration the statistical nature of the phenomenon.

The best mechanism has been proposed by Orowan (reference 13), but the statistical aspect of the theory of fatigue is in need of much work to put it on a quantitative basis. The factors which have been previously discussed schematically and qualitatively should be treated quantitatively. This treatment, in the ultimate, would involve a knowledge of the distribution of inclusions from measurement, calculations of the stress-concentration factors for the imperfections, and numerical probabilities for the presence of effective imperfections in the critical region of the specimen at maximum nominal stress. Although difficult, this approach combined with Orowan's mechanism might be fruitful.

There have been a few attempts to develop theories of fracture and fatigue on a statistical basis. These theories, however, have not considered the statistical variation in strength but have derived the mean-strength relationships by statistical treatments. These theories will be critically reviewed.

Weibull (reference 34) derived a statistical theory for fracture in Griffith solids. His treatment applied to brittle materials such as glass which are isotropic and obey Hooke's law up to fracture (i.e., no plasticity). He chose a random distribution of flaws and assumed that the probability that fracture occurs in any one volume at a given stress is independent of the probability of fracture in any other given volume at the same stress. The theory has been successfully applied to glass, but it is not intended for use in the problem of fracture in metals.

Fisher and Hollomon (reference 35) attempted to develop a statistical theory for fracture in metals on the basis of essentially the same assumptions made by Weibull. As Zener (reference 14) has pointed out, the assumption that the probabilities of fracture in given volumes at the same stress are independent does not apply to metals. Unlike glass, metals undergo plastic deformation with which fracture is associated. When slip occurs in a volume, the probability of fracture in the region in front of the slip band is higher than that for the other elements. Thus, the assumption of independent probabilities is invalid for materials which deform plastically and this statistical theory for fracture in metals is of limited value.

Freudenthal (reference 36) has proposed a statistical theory based on the belief that fatigue failure is not inseparably associated with plastic slip (in direct opposition to experimental evidence and the modern theories of fracture in metals). He assumes that fatigue is a large-scale expression of the progressive destruction of atomic cohesive bonds and that plastic slip and strain-hardening are essential features only insofar as they modify the intensity and rate of bond destruction. A statistical treatment was made involving the number and strength of cohesive bonds and the number and intensity of load repetitions. By applying the theory of probability, Freudenthal deduced relationships for the S-N curve and other fatigue phenomena.

This theory cannot be considered acceptable. As pointed out above, the probability of failure under a given stress at the outset no longer holds after this stress has been applied. Metals undergo plastic deformation with which fracture is associated and, once slip has occurred in a volume, the probability of fracture in this region is increased. Thus, Freudenthal's theory cannot be applied to materials which deform plastically. As a point of interest, it might be mentioned that brittle nonmetallics such as porcelain or glass exhibit static fatigue rather than dynamic fatigue as found with metals. That is, in the fatigue of brittle nonmetallics, the time under stress rather than the number of stress cycles determines failure. Thus, Freudenthal's theory which is based on assumptions which might be valid for brittle nonmetallics does not apply to this case either.

Afanesiev (reference 37) proposed a statistical theory of fatigue based on the averaging of stress-strain relationships within the individual grains of a polycrystalline metal. It is assumed that the yield strength in all grains is equal in the direction of the acting force, but the stress differs because of crystal anisotropy and inhomogeneities. (The same results are obtained if equal stress but different yield strengths are assumed.) Assuming a frequency distribution of the stresses in the crystals and that strain-hardening is linearly related to the plastic deformation, the increase in flow stress as a function of the number of cycles is calculated. A crack occurs when the flow stress reaches the value of the strength of the metal. Fatigue failure requires the formation of a number of cracks in adjacent grains and the probability of occurrence of this condition is calculated yielding expressions for the fatigue strength as a function of the number of cycles.

The method of attack in this theory is quite interesting, but the assumptions made throw some doubt on its validity in its present form. The assumptions of a particular frequency distribution of stress, a linear relationship between strain-hardening and plastic deformation, and the necessity of producing more than one crack for fatigue failure are made without justification. In addition, the parameters in the probability function were taken from a static-tension stress-strain curve which does not apply to the condition of dynamic cyclic stressing.

It is evident that a purely statistical approach has not led to a satisfactory theory for the fatigue of metals. Undoubtedly, the mechanism by which metals fail in fatigue is governed by the fundamental laws of plastic deformation and fracture; it is only through the application of these laws that a successful theory of fatigue can be derived (which up to the present time has been best done by Orowan). The statistical aspect of fatigue arises from the presence (or scarcity) of imperfections in the metal, which modify the conditions of flow and fracture and introduce a variation in properties.

Size effect.- The size effect refers to the observation that the endurance limit in rotating-cantilever tests increases with decrease in specimen diameter. Moore and Morkovin (references 38 to 41) conducted rotating tests on several steels in both the notched and unnotched conditions and found a decrease in endurance limit with increase in specimen diameter up to a certain diameter (1/2 to 1 in.) above which (up to 3 in.) the endurance limit was not affected. Von Rajakovics (reference 42) found a similar trend in the alternating bending strength of aluminum alloys as did Donlan and Hanley (reference 43) with SAE 4340. Horger and Neifert (reference 44) performed rotating tests on axles with diameters as large as $6\frac{1}{2}$ inches and observed a marked decrease in the endurance limit with increase in specimen diameter.

The usual explanation of this effect is based on the "weakest link" theory. According to this view, the probability of obtaining a serious internal flaw or stress-raiser in the specimen increases with increase in the volume of the specimen and therefore lower fatigue properties are obtained with larger-diameter specimens. Some of the work in the present investigation has substantiated this concept, but there is an additional factor in this problem which has been overlooked by all but a few investigators.

The rotating-cantilever fatigue test, with which most of the size-effect studies have been made, produces a stress gradient across the diameter of the specimen since one side is in pure tension while the opposite side is in pure compression. Thus tests of this kind on specimens of different diameters involve correspondingly different stress gradients. Phillips (reference 45), Föppl (reference 46), and Buchmann (references 47 and 48) have studied the effect of this factor on the size effect and concluded that with a steeper stress gradient (smaller-diameter specimen) the underlying fibers have a greater opportunity to aid the outermost fibers in supporting the stress. Buchmann has conducted axial push-pull fatigue tests (which eliminate the stress gradient across the section) and found only a small increase in the fatigue strength of steels and aluminum with decrease in specimen diameter. He also showed that the fatigue properties under rotating bending of specimens with increased diameter approach the strength obtained under reversed axial loading as a lower limit.

There have been no studies of the effect of specimen size on the statistical variation in fatigue properties; all the work reviewed above has dealt with the average properties. In the previous discussion, however, the size effect was briefly considered in the interpretation of the dependence of the statistical variation in fatigue life on the stress level in rotating and torsional tests. In the proposed hypothesis it was pointed out that the maximum nominal stress is limited to a small region in the R. R. Moore and flexural test, whereas in torsion it exists along the entire gage length and this state of affairs influences the statistical behavior. An extension of these ideas can be used to explain the experimental observations of the effect of specimen size on the average endurance limit. In the present discussion, the statistical variation about the mean will be ignored and only the mean will be considered.

The first and obvious point to be made is that the endurance limit for a material with many imperfections is lower than that for the same material with extremely few imperfections. Thus the endurance limit of a material is strongly influenced by the presence or absence of imperfections. In the axial push-pull test the maximum nominal stress is uniform along the gage length and the cross section and therefore any point in the volume of the specimen can be the locus of crack initiation. Changes in the diameter of this type of specimen have little influence on the probability of occurrence of an effective imperfection in the test volume, because in this case the critical volume is vast at all diameters and the probability of encountering an active imperfection is very high. Consequently, the endurance limit in push-pull tests does not increase appreciably with decrease in specimen diameter and further the endurance limit obtained is relatively low because it is certain that effective imperfections are encountered.

On the other hand, in the R. R. Moore test, only a small region is subjected to maximum applied stress. Aside from the differences in stress gradient, the smaller-diameter specimens have a smaller critical-stress volume which decreases the probability of the existence of a severe imperfection in this volume. Thus a decrease in diameter produces an increase in endurance limit. Upon increasing the diameter, the critical volume increases (as does the probability of imperfection occurrence) to a size effectively the same as that existing in push-pull tests and the same endurance limits are obtained in the two types of test.

S-N diagram.- In the discussion of the work presented in part I, the need for an improved method of reporting S-N data was pointed out and a possible method was suggested. This method had been applied to a number of sets of fatigue data obtained by Wallgren (reference 21) with some success as illustrated by an example.

The S-N data obtained in the present investigation have also been treated in this manner, but with less success. The expected straight line was obtained, but this line did not pass through the tensile-strength point. The reason for this behavior may well lie in the nature of the tests conducted in this research investigation. Wallgren's data, to which the method applied quite well, were obtained from axial tension-compression tests and quite logically the tensile strength can be used in the analysis. The present fatigue experiments, however, were performed in flexure and perhaps, therefore, the fracture strength in bending rather than the tensile strength should be used as the upper limit of the S-N curve. Similarly, it would seem likely that in representing the complete S-N diagram obtained in torsional fatigue tests, the fracture strength in torsion rather than the tensile strength is the true upper limit in the diagram. Since data for the fracture strength in bending were not available for the materials studied in this research, this possibility could not be investigated quantitatively. It is a safe assumption, however, that the fracture strength in bending is higher than the tensile strength of these materials. The effect of using a higher strength value as the upper limit of the S-N curve is to decrease the slope of the straight line obtained in the proposed method of plotting the diagram. Therefore, by this modification, the slopes of the straight-line S-N plots shown in the previous figures would be altered toward better agreement with prediction (i.e., a straight line from the endurance limit to the fracture strength at 1/4 cycle).

Thus, in its present form, the proposed method of treating the entire S-N diagram has met with limited success. Further development and modification may yield a more satisfactory method of representing the fatigue data obtained from tests other than the axial tension-compression type.

In connection with the Wöhler diagram, it is important to note that many of the existing theories of fatigue predict the form of the S-N curve which has been believed to exist (a simple concave upward curve). As shown in the present research, however, this mean curve is not simple but has a point of inflection and bends toward the stress axis at higher stress and shorter fatigue life. Thus the existing theories must be extended or modified to account for this behavior.

Understressing effect.- The experimental results have shown that understressed specimens have, on the average, a longer life than (or at worst, the same life as) the virgin specimens at a stress in the fracture range. This observation proves first of all that the behavior described by Jacques (reference 49) does not generally exist. Jacques has reported that measurable fatigue cracks (up to 0.01 in. in depth) develop and grow with increase in number of cycles in specimens tested at and below the endurance-limit stress. If this were generally true, it is to be expected that understressing produces microcracks which would decrease the subsequent fatigue life relative to the virgin life at an elevated stress. The present experiments as well as others reported in the literature show that the contrary occurs.

The usual explanation of the beneficial effect of understressing is that the cycles of stress at or below the endurance limit mildly cold-work and strengthen the material and thus extend the subsequent life at higher stresses (reference 50). This explanation is strengthened by reports that the fatigue life can be extended very appreciably by training or coaxing, but in view of the experimental results obtained in this investigation it is proposed that the understressing effect may be interpreted, in part, as a statistical phenomenon based on selectivity.

Consider a distribution in quality of virgin specimens such as that shown in figure 41. Upon testing these specimens at the mean endurance limit $\overline{S}$, 50 percent of them fail while the others run out. Those which fail represent poorer than average quality while those which do not fail are of better than average quality (see fig. 41). If these run-out specimens are now tested at an elevated stress, the mean fatigue life should be longer than that obtained for the virgin material. The latter includes specimens of all qualities while those which were understressed have been preselected.

Similarly, if specimens are tested at a stress somewhat higher than the mean endurance limit $\bar{S} + \Delta S$, a certain number of failures and nonfailures are obtained. (The exact proportion depends upon the level of stress above the mean endurance limit and the standard deviation about this mean.) In this case, the percentage of run-out specimens is less than 50 percent and, since the stress is more severe, the specimens which do not fail are of superior quality (see fig. 41). The argument can also be applied to specimens understressed at a level below the mean endurance limit $\bar{S} - \Delta S$ to show that subsequent mean life is slightly better than the virgin mean life.

Thus run-out specimens at $\bar{S} + \Delta S$ are, on the average, superior to run-out specimens at $\bar{S}$, which in turn are of higher quality than the average of nonfailure specimens at $\bar{S} - \Delta S$, which are slightly better on the average than the virgin materials; these differences should be reflected in the relative fatigue lives obtained in retest at a stress in the fracture range.

This interpretation applies quite nicely to the results obtained in the understressing experiments. The mean fatigue lives $\bar{N}$ for specimens understressed above $\bar{S} + \Delta S$, at $\bar{S}$, and below $\bar{S} - \Delta S$ fall in the following order: $\bar{N}$ for $\bar{S} + \Delta S$ is greater than $\bar{N}$ for $\bar{S}$ and $\bar{N}$ for $\bar{S} - \Delta S$ is greater than $\bar{N}$ for virgin specimens. This behavior indicates that the understressing effect may well be partially based on selectivity and the statistical variation in fatigue properties. The question as to the relative importance of cold-work and selectivity could be answered by further experiment.

A purely mathematical approach to the understressing effect can be made by assuming that the behavior is based entirely on selectivity. It is possible to calculate the mean life expected from specimens which represent a certain quality of the parent population. For example, those specimens which do not fail when tested at $\bar{S}$ (the mean endurance limit) represent better than average quality; and, since 50 percent of the specimens do not fail at $\bar{S}$, the test has selected the upper 50 percent of the virgin population. The expected mean life of these specimens when tested at a stress in the fracture range can be calculated from straightforward statistics. The relationship is simply:

$$X = m + u\sigma$$

where

X mean fatigue life at fracture stress of specimens which did not fail in understress, cycles

m mean fatigue life for virgin specimens at fracture stress, cycles

σ	standard deviation in fatigue life for virgin specimens
u	level corresponding to mean life of understressed bars in retest at fracture stress (tables of u are available in handbooks)

In the case of Armco iron specimens understressed at $\overline{S}$, the calculation is as follows (remembering that fatigue life has a normal distribution for log N):

$$m = \overline{\log N} = 5.7691 \text{ or } 5.876 \times 10^5 \text{ cycles}$$

$$u = 0.67$$

$$\sigma = 0.164$$

therefore

$$X = m + u\sigma$$

$$= 5.7691 + 0.67(0.164)$$

$$= 5.8790 \text{ or } 7.56 \times 10^5 \text{ cycles}$$

The value of this mean life obtained from experiment was 8.49×10^5 cycles.

Similar calculations made for other understressing levels and materials are summarized in the following table:

Material	Understress (psi)	Mean fatigue life in retest at fracture stress	
		Predicted from selectivity (cycles)	Experimental result (cycles)
Armco iron	32,800 29,820 26,840	1.09×10^6 7.56×10^5 6.17	9.70×10^5 8.29 8.48
SAE 4340(Q&S)	65,150 62,430 59,720	6.55×10^5 4.71 3.52	8.31×10^5 3.81 3.27
SAE 4340(Q&T)	100,420 97,700 94,990	1.92×10^5 1.74 1.57	2.73×10^5 2.51 2.15

The numerical agreement here is only fair which is as much as can be expected in view of the fact that the number of understressed specimens used was necessarily limited. Agreement does exist between prediction and experiment in trend and magnitude. If it were found experimentally that the mean subsequent fatigue life of understressed specimens was far greater than that predicted on the basis of selectivity, it could be concluded that cold-work is the predominant factor in the understressing phenomenon. The above results indicate that this is not true and that the effect is, to at least some extent, based on selectivity.

The above results and their interpretation give the subjects of overstressing and understressing a whole new aspect. For example, Bennett (reference 19) has reported that the damage produced by overstress can be recovered by subsequent understress. This conclusion was based on the experimental observation that a specimen of SAE 4130 overstressed at 48,000 psi, then understressed, and finally retested at the original stress of 48,000 psi did not fail. The endurance limit of the virgin material was 46,000 psi.

In view of the statistical nature of fatigue properties, the above conclusion can be questioned. It can be shown by statistical calculations that there was a certain probability that the original specimen would not have failed if the test had been continued at the initial level of overstress. This calculation will be illustrated.

The relative standard deviation in the endurance limit of a steel can be conservatively estimated as 3.5 percent. Thus, $\frac{\sigma}{\overline{S}} \times 100 = 3.5$ percent, where σ is the standard deviation in endurance limit and $\overline{S}$ is the mean endurance limit. The present material (SAE 4130) had an endurance limit of 46,000 psi, which can be taken as the mean value. Therefore, from the above equation, the standard deviation in the endurance limit is: $\sigma = (3.5 \text{ percent})(46,000) = 1610$ psi. This value of σ can be used to calculate the probabilities of failure and nonfailure at 48,000 psi. Transforming into probits gives the following:

$$Y = 5 + \frac{1}{\sigma}(S - \overline{S})$$

$$= 5 + \frac{1}{1610}(48,000 - 46,000)$$

$$= 6.24$$

The value obtained is the probit value at a stress of 48,000 psi and corresponds to 80 percent in the probit-percentage tables (reference 18). This means that 80 percent of the specimens tested at 48,000 psi should fail, while 20 percent can be expected to sustain this applied cyclic stress without failure. Thus, calculation indicates that one out of every five specimens of this material tested at 48,000 psi should not fail and that the conclusion based on a single experimental observation can be seriously in error. The effect attributed to a beneficial understressing subsequent to the overstress could be due to the fact that the specimen used was one which would not have broken at the overstress level anyway; the chance of obtaining such a specimen in the experiment cited is estimated as one out of five.

Obviously, many conclusions with regard to the fatigue of metals which have been generally accepted in the past must now be re-examined from a viewpoint which includes the statistical nature of fatigue properties.

Engineering application.- In the design for finite life of parts subjected to fatigue stresses, the engineer should take into account the statistical variation in fatigue life to minimize premature failures. It is doubtful that many companies will go to the trouble and expense of determining experimentally the statistics of the fatigue properties of the metals involved, so some approximate rule should be provided. It has been found that the relative standard deviation in fatigue life for steels is roughly 3 percent $\left(\text{i.e.,}\quad V = \frac{\sigma}{\overline{\log N}} \times 100 = 3\right)$. Thus, if an approximate S-N curve is known from a dozen or more fatigue tests, an estimate of the standard deviation in fatigue life can be obtained by taking 3 percent of the logarithm of the number of cycles to failure. There is no precise substitute for the complete statistical data, but this procedure could serve as a rough guide to the designer.

Similarly, in the case of the endurance limit, a relative-standard-deviation value of 4 percent $\left(V = \frac{\sigma}{\overline{S}} \times 100 = 4\right)$ could be used. Knowing the approximate value of the endurance limit from an S-N curve determined in the usual manner, the standard deviation from this value could be estimated by multiplying by 4 percent. Using this standard deviation, the probability of failure at stresses below the endurance limit could be calculated and the proper factor of safety chosen.

CONCLUSIONS

Statistics of the fatigue-fracture curves and endurance limits for a variety of materials were obtained from extensive fatigue tests and from a critical review of the literature. The results were analyzed to determine the relative effects of some metallurgical factors on the statistical nature of fatigue properties.

In addition, a number of related problems were studied including the dependence of statistical variations in fatigue life on stress level in the fracture range, location of crack initiation, size effect, understressing effect, and a form and possible method of plotting the S-N diagram. An approximate method of predicting premature failure in steel in the absence of statistical data was also determined.

As a result of this study and research, the following conclusions may be drawn:

1. Both the fatigue life and endurance limit are subject to marked variability and, therefore, these quantities cannot be stated as exact values, but must be represented statistically.

2. In view of the statistical nature of fatigue properties, the design engineer must, in choosing a factor of safety, recognize the fact that some failures can occur prematurely or even at stresses below the normally determined endurance limit.

3. The S-N diagram does not follow the simple curve which is usually drawn but has a point of inflection and bends toward the stress axis at shorter life.

4. Statistical analysis of the measurements of the location of fatigue crack initiation provide further evidence of the inhomogeneous nature of the steel and its influence on the statistical behavior in fatigue.

5. The dispersion in fatigue life in rotating and vibrating tests increases with decrease in stress in the fracture range. In torsional fatigue, however, the dispersion is independent of the level of the stress.

6. The variability in fatigue properties depends upon the metal and is influenced by metallurgical factors.

7. The statistical variation in the fatigue life of nonferrous metals (e.g., copper and aluminum) is less than that for iron or steels, probably because the former materials have fewer inclusions and inhomogeneities.

8. Of the many factors which can influence the statistical behavior of fatigue properties, inclusions are the most important. Two steels of the same chemical composition (e.g., two heats of SAE 4340) but with widely different inclusion ratings have correspondingly different variabilities.

9. The effect of differences in composition and microstructure of iron and steels on the degree of dispersion is overshadowed by the inclusion contents of the materials. A steel of given composition can have greater or less variability than iron depending upon whether its inclusion rating is relatively high or low.

10. At a given inclusion rating, the material with higher ductility yields less scatter in the endurance limit, but the dispersion in fatigue life is essentially the same.

11. The understressing effect may, in part if not wholly, be interpreted as a statistical phenomenon. Experimental results conform to theoretical predictions based purely on selectivity.

12. The size effect can be explained on a statistical basis.

Carnegie Institute of Technology
Pittsburgh, Pa., May 21, 1951

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TABLE I

RESULTS OF STATISTICAL CALCULATIONS BASED ON RAVILLY'S DATA (REFERENCE 6)

[$\overline{\log N}$, mean of logs of life values; $\overline{N}$, mean life, cycles; σ, unbiased standard deviation; $\sigma_{\overline{\log N}}$, standard estimate of error; V, relative standard deviation, percent]

(a) Cold-worked steel.

Stress level	$\overline{\log N}$	$\overline{N}$ (cycles)	σ	$\sigma_{\overline{\log N}}$	V (percent)
First	4.95868	90,925	0.4371	0.0977	8.814
Second	5.63674	433,250	.3353	.7497	5.948
Average	-------	-------	.3896	------	7.381

(b) Annealed steel.

Stress level	$\overline{\log N}$	$\overline{N}$ (cycles)	σ	$\sigma_{\overline{\log N}}$	V (percent)
First	4.99680	99,265	0.1812	0.0405	3.626
Second	5.65839	455,400	.1889	.0422	3.338
Average	-------	-------	.1851	------	3.482

(c) Annealed Armco iron (unsorted).

Stress (kg/mm^2)	$\overline{\log N}$	$\overline{N}$ (cycles)	σ	$\sigma_{\overline{\log N}}$	V (percent)
13.0	5.06476	116,080	0.1596	0.03568	3.151
11.5	5.40834	256,060	.1689	.0377	3.122
10.5	5.99171	981,100	.1311	.0293	2.188
Average	-------	-------	.1541	-------	2.820

TABLE I.- Continued

RESULTS OF STATISTICAL CALCULATIONS BASED ON RAVILLY'S

DATA (REFERENCE 6) - Continued

(d) Annealed Armco iron (presorted).

Stress (kg/mm^2)	$\overline{\log N}$	$\overline{N}$ (cycles)	σ	$\sigma_{\overline{\log N}}$	V (percent)
31.0	4.11268	12,962	0.03211	0.00718	0.780
26.0	4.22988	16,977	.02734	.00611	.646
21.5	4.42353	26,517	.02294	.00513	.518
18.0	4.61583	41,290	.02919	.00652	.632
14.5	4.85753	72,033	.02554	.00571	.525
13.0	5.09362	124,060	.02595	.00580	.509
12.0	5.27894	190,100	.02384	.00533	.451
11.75	5.43991	275,310	.03893	.00870	.715
11.0	5.63216	428,700	.02554	.00571	.453
10.7	6.01125	1,026,250	.02828	.00632	.470
Average	-------	---------	.02832	-------	.569

(e) Annealed nickel (presorted).

Stress (kg/mm^2)	$\overline{\log N}$	$\overline{N}$ (cycles)	σ	$\sigma_{\overline{\log N}}$	V (percent)
56.0	4.14467	13,953	0.04147	0.00951	1.000
49.0	4.33214	21,485	.03095	.00692	.714
44.5	4.47014	29,521	.04328	.00967	.968
37.5	4.69945	50,054	.01123	.00251	.238
33.0	4.85572	71,731	.03756	.00839	.773
30.0	4.99631	99,154	.03960	.00885	.792
25.5	5.18663	153,690	.05040	.01127	.971
21.5	5.40029	251,360	.04193	.00937	.776
18.0	5.67544	473,630	.03671	.00820	.646
15.5	6.08865	1,226,500	.04436	.00991	.728
Average	-------	---------	.03908	-------	.760

NACA

TABLE I.- Concluded

RESULTS OF STATISTICAL CALCULATIONS BASED ON RAVILLY'S

DATA (REFERENCE 6) - Concluded

(f) Annealed steel (presorted).

Stress (kg/mm^2)	$\overline{\log N}$	$\overline{N}$ (cycles)	σ	$\frac{\sigma}{\overline{\log N}}$	V (percent)
51.0	4.81222	64,897	0.04483	0.01000	0.931
43.75	4.91119	81,506	.04364	.00975	.888
38.75	5.00345	100,800	.04436	.00991	.886
29.5	5.18268	152,290	.04328	.00967	.835
23.0	5.31848	208,200	.03111	.00695	.584
18.0	5.47979	301,850	.04909	.01097	.895
14.25	5.70197	503,470	.03244	.00725	.568
12.5	5.86293	729,330	[a].78914	[a].17646	[a]13.45
Average	-------	-------	.04172	------	.798

[a]Questionable values.

(g) Annealed copper (unsorted).

Stress (kg/mm^2)	$\overline{\log N}$	$\overline{N}$ (cycles)	σ	$\frac{\sigma}{\overline{\log N}}$	V (percent)
27.0	4.03823	10,920	0.14647	0.03275	3.627
22.5	4.20321	15,967	.1253	.02800	2.98
19.0	4.32805	21,284	.08772	.01960	2.026
16.0	4.50477	31,972	.12204	.02728	2.686
13.5	4.71066	51,364	.09469	.02117	2.010
11.75	4.83403	68,238	.09536	.02132	1.972
10.0	5.04901	111,950	.07772	.01737	1.539
9.25	5.23843	173,150	.09442	.02111	1.802
8.25	5.49007	309,080	.09346	.02089	1.702
7.25	5.87667	752,780	.1541	.03445	2.622
Average	-------	-------	.1124	-------	2.296

(h) Annealed aluminum (unsorted).

Stress (kg/mm^2)	$\overline{\log N}$	$\overline{N}$ (cycles)	σ	$\frac{\sigma}{\overline{\log N}}$	V (percent)
30.0	3.92328	8,380.7	0.09072	0.02028	2.312
22.5	3.98474	9,654.7	.1172	.0262	2.941
18.0	4.10769	12,814.0	.1154	.0258	2.809
13.75	4.23805	17,300	.1572	.0351	3.709
10.75	4.36346	23,092	.1131	.02529	2.591
8.5	4.57964	37,988	.1275	.0285	2.784
7.5	4.89601	78,707	.1323	.02958	2.702
5.75	5.32108	209,450	.1240	.0277	2.330
5.5	5.71952	524,225	.1463	.0327	2.557
5.25	6.05450	1,133,700	.06996	.0156	1.155
Average	-------	-----------	.1218	------	2.589

NACA

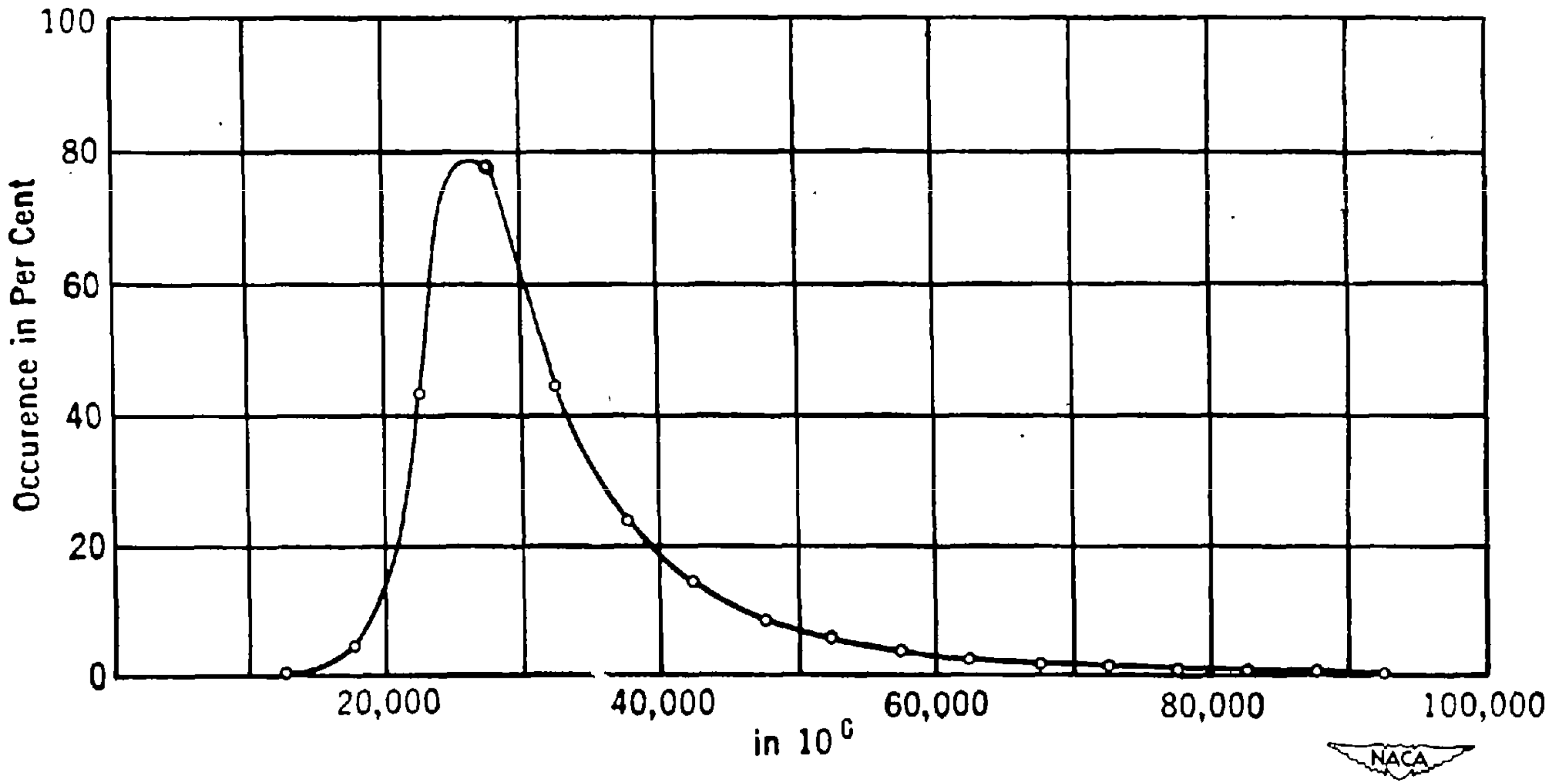


Figure 1.- Frequency distribution in fatigue life of 200 steel specimens at 45,500 psi. (Figure taken from reference 2.)

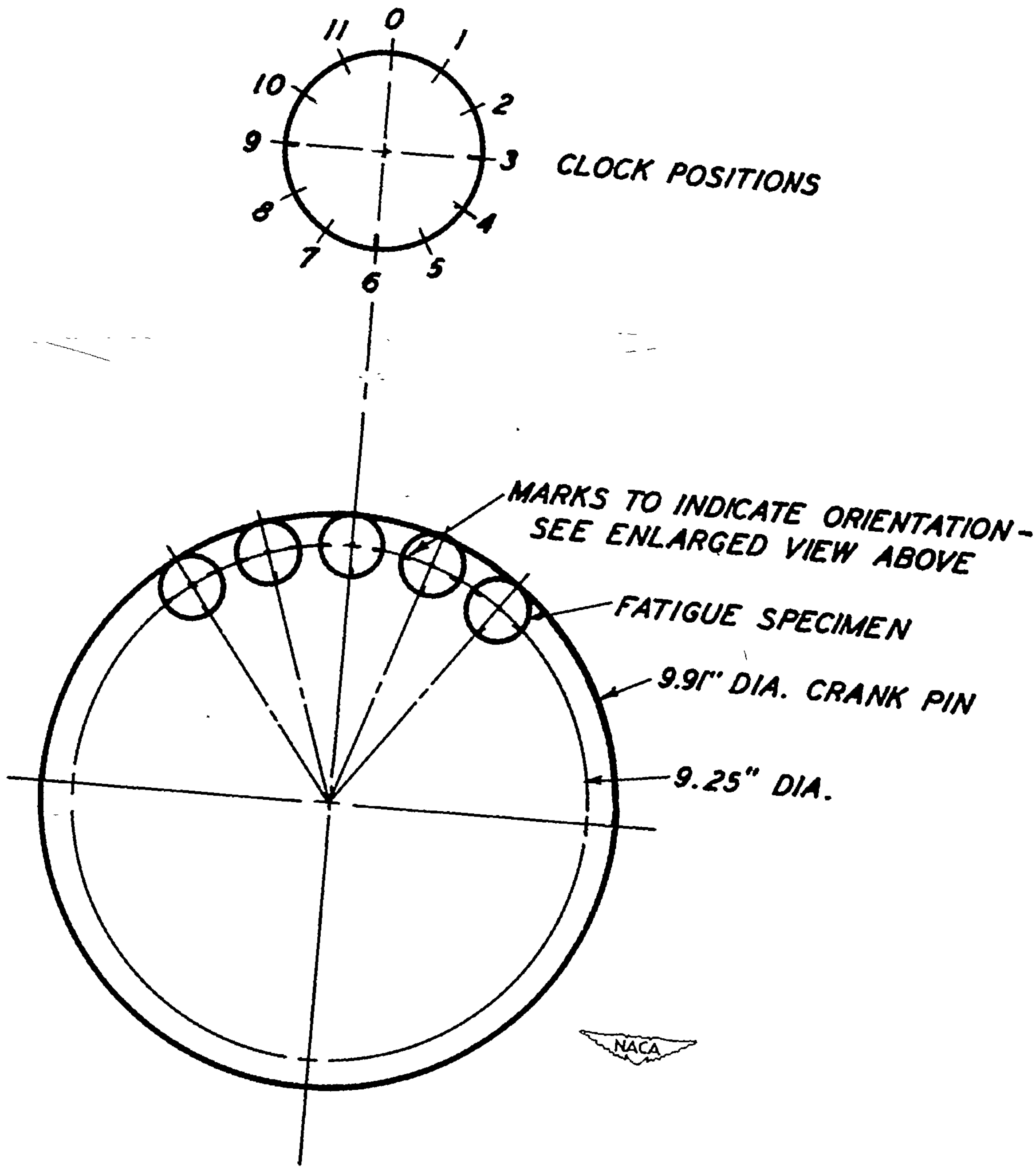


Figure 2.- Location of specimens in crankpin and clock notation based on end marks.

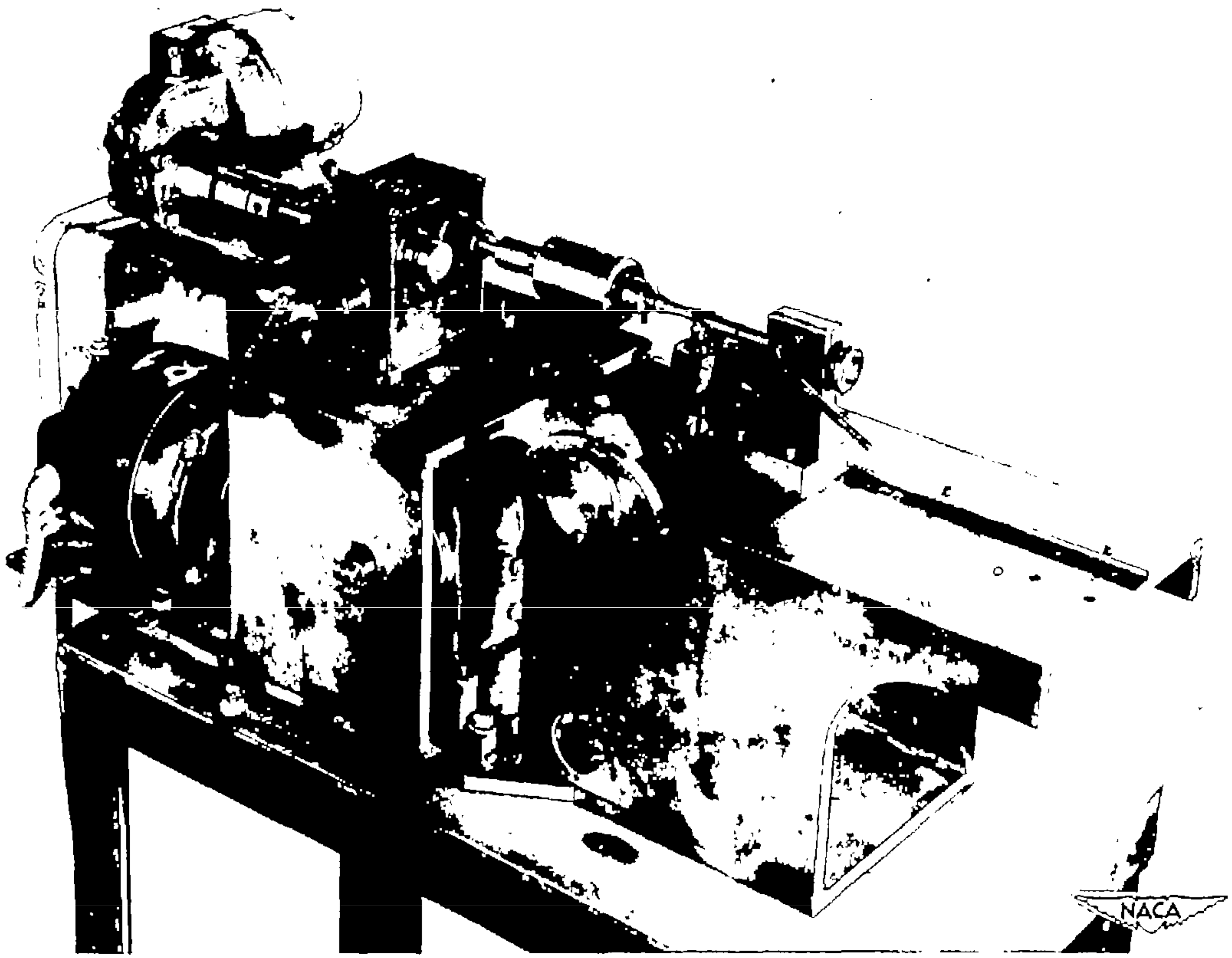


Figure 3.- Machine for honing specimens longitudinally.

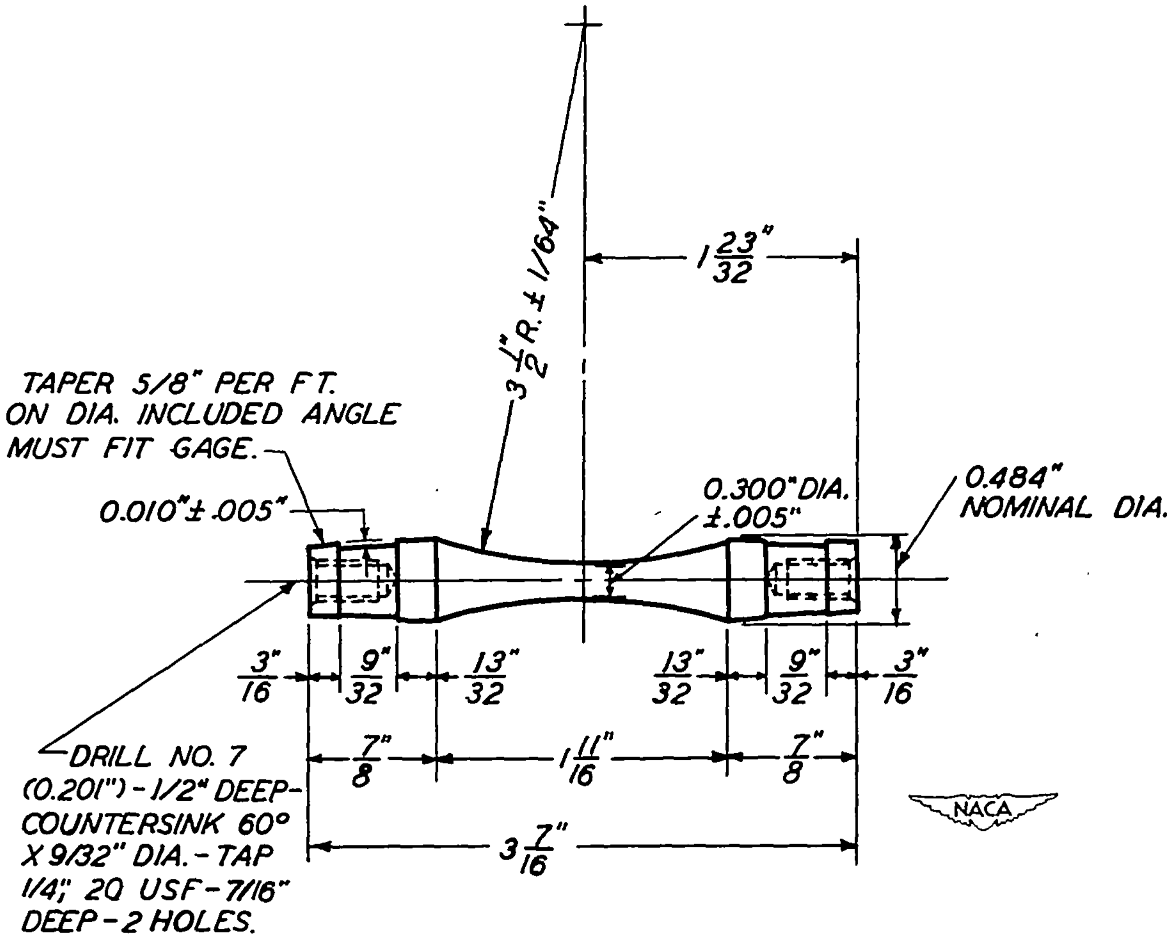


Figure 4.- Rotating-cantilever fatigue test specimen. Nominal dimensions are shown; ±1/64-inch tolerance is allowed on all longitudinal dimensions.

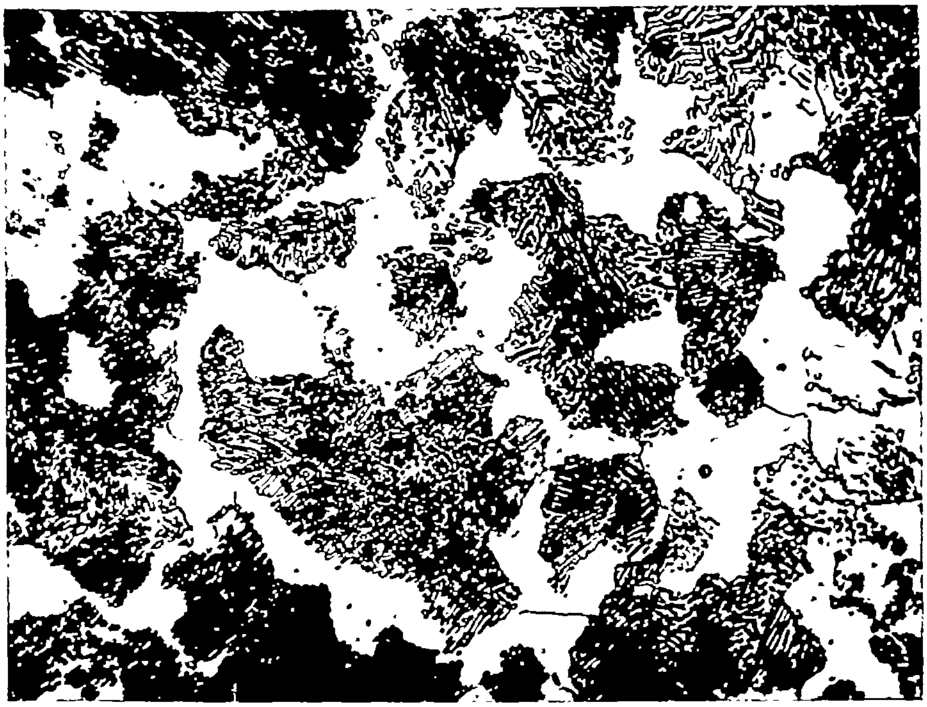


Figure 5.- Microstructure of normalized and tempered SAE 1050. 1000X; nital etch.

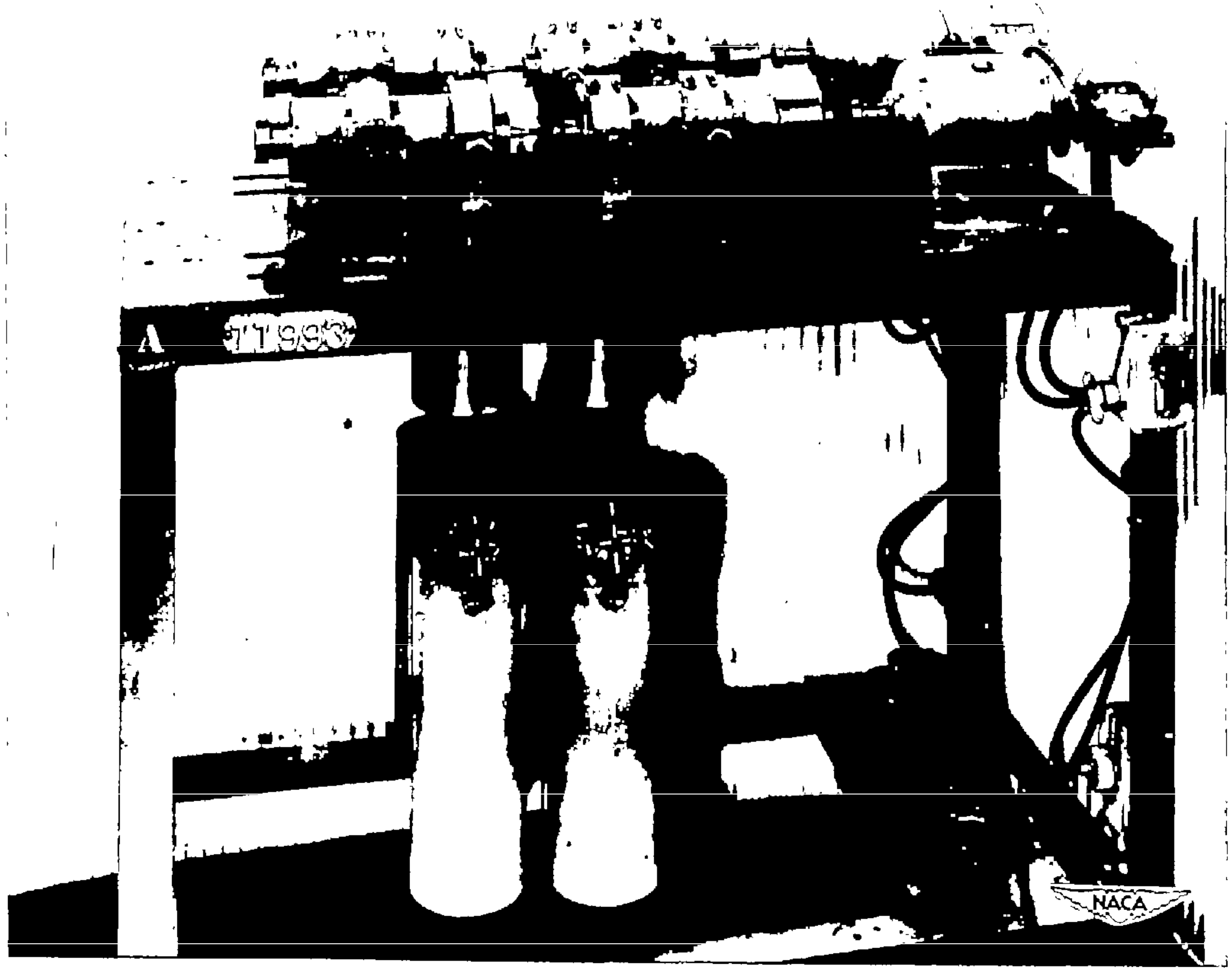


Figure 6.- Rotating-cantilever fatigue machine.

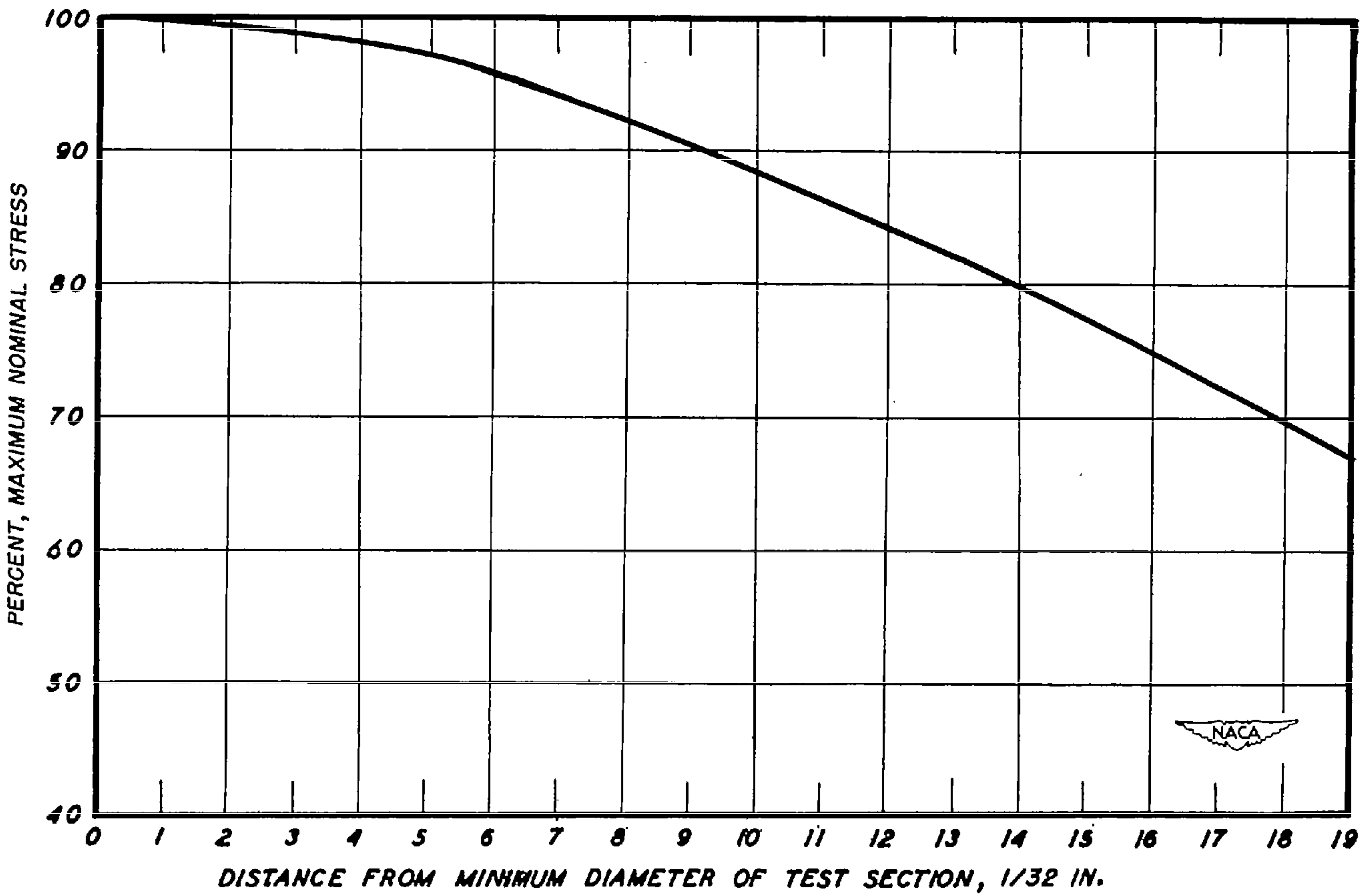


Figure 7.- Stress correction curve.

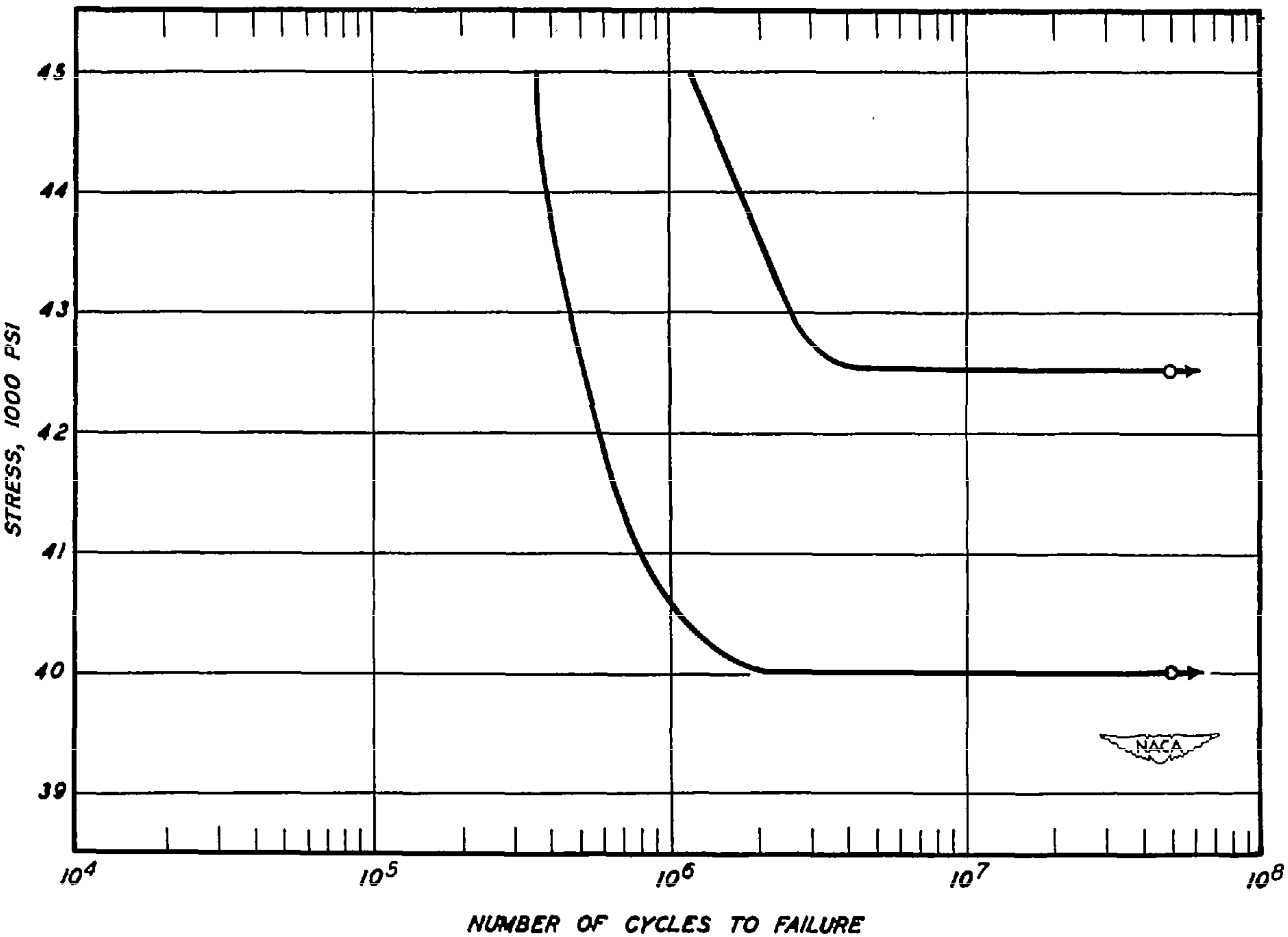


Figure 8.- Scatter band of SAE 1050 fatigue data.

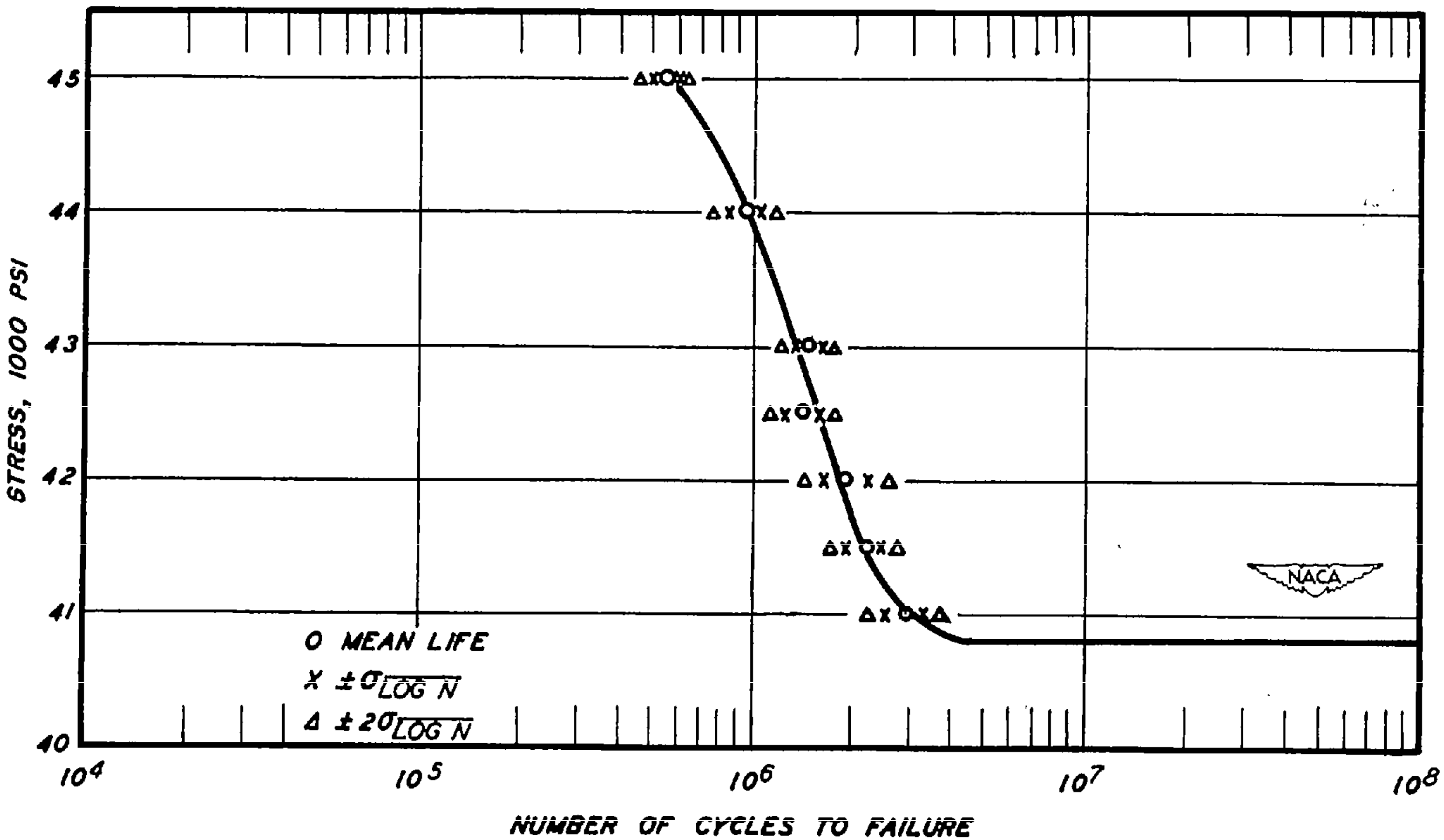


Figure 9.- Mean fatigue life for samples of SAE 1050 and statistically possible values of mean for universe.

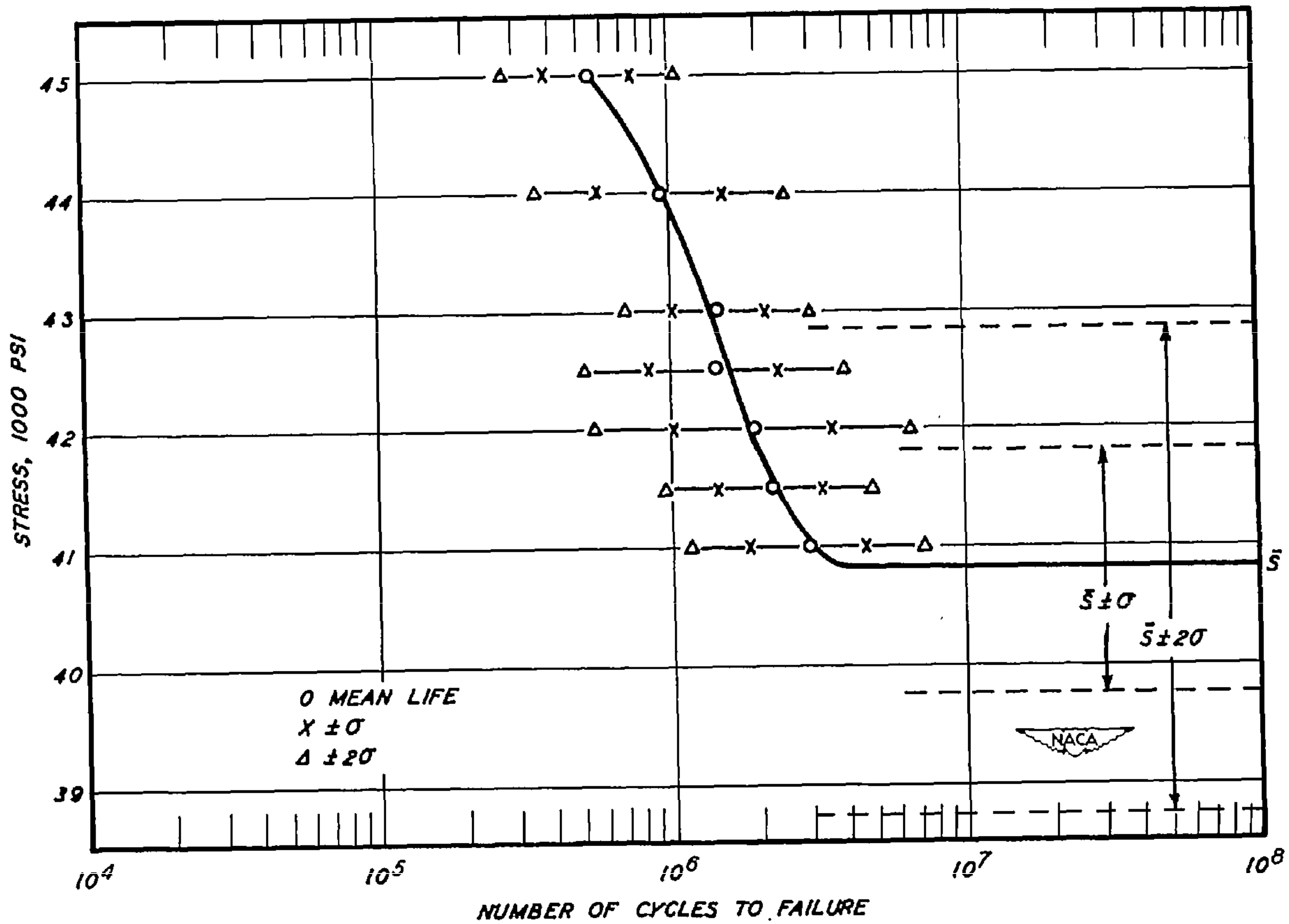


Figure 10.- Statistical variation in fatigue life and endurance limit of SAE 1050.

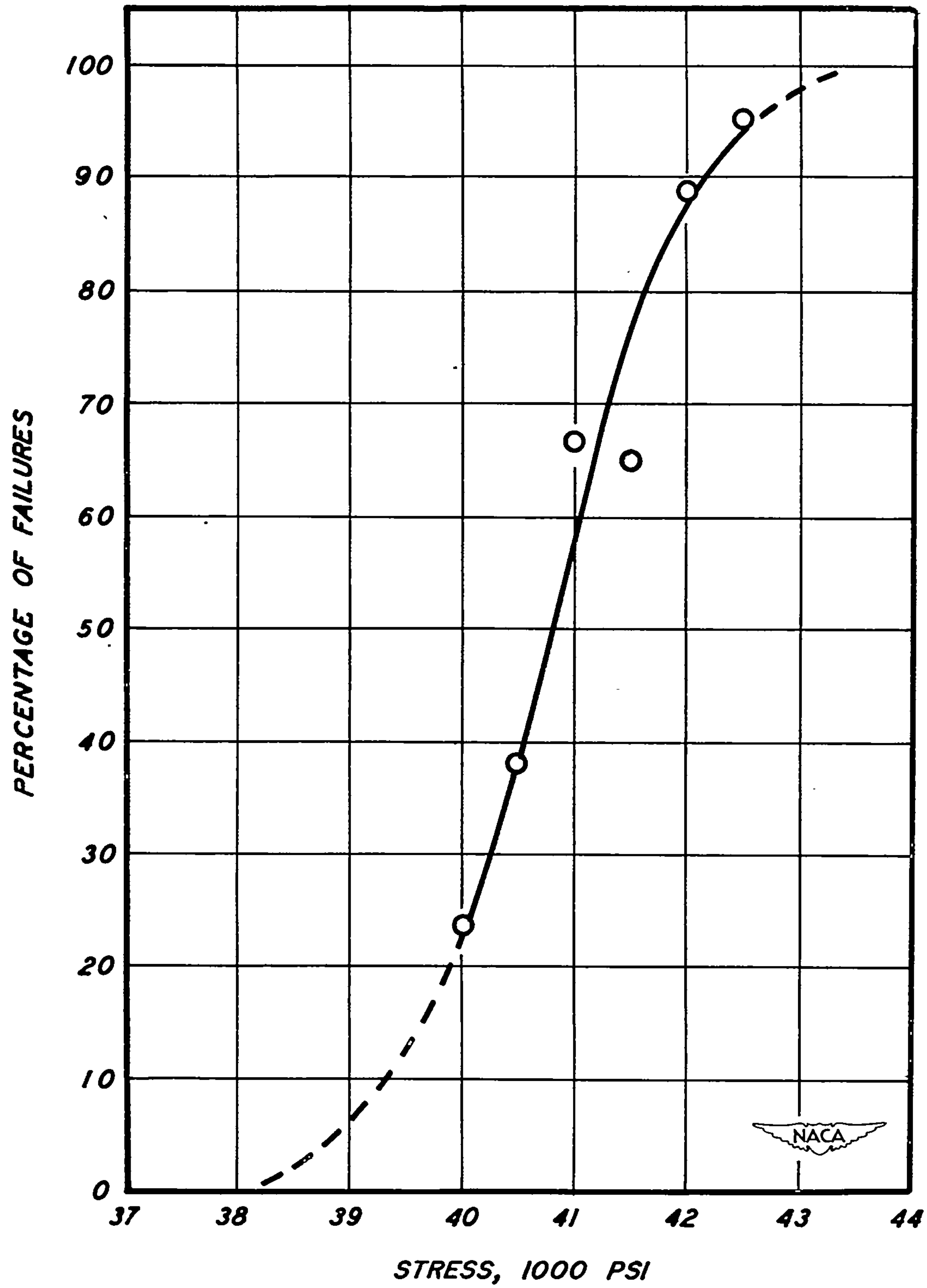


Figure 11.- Mortality curve for endurance range of SAE 1050 in linear coordinates.

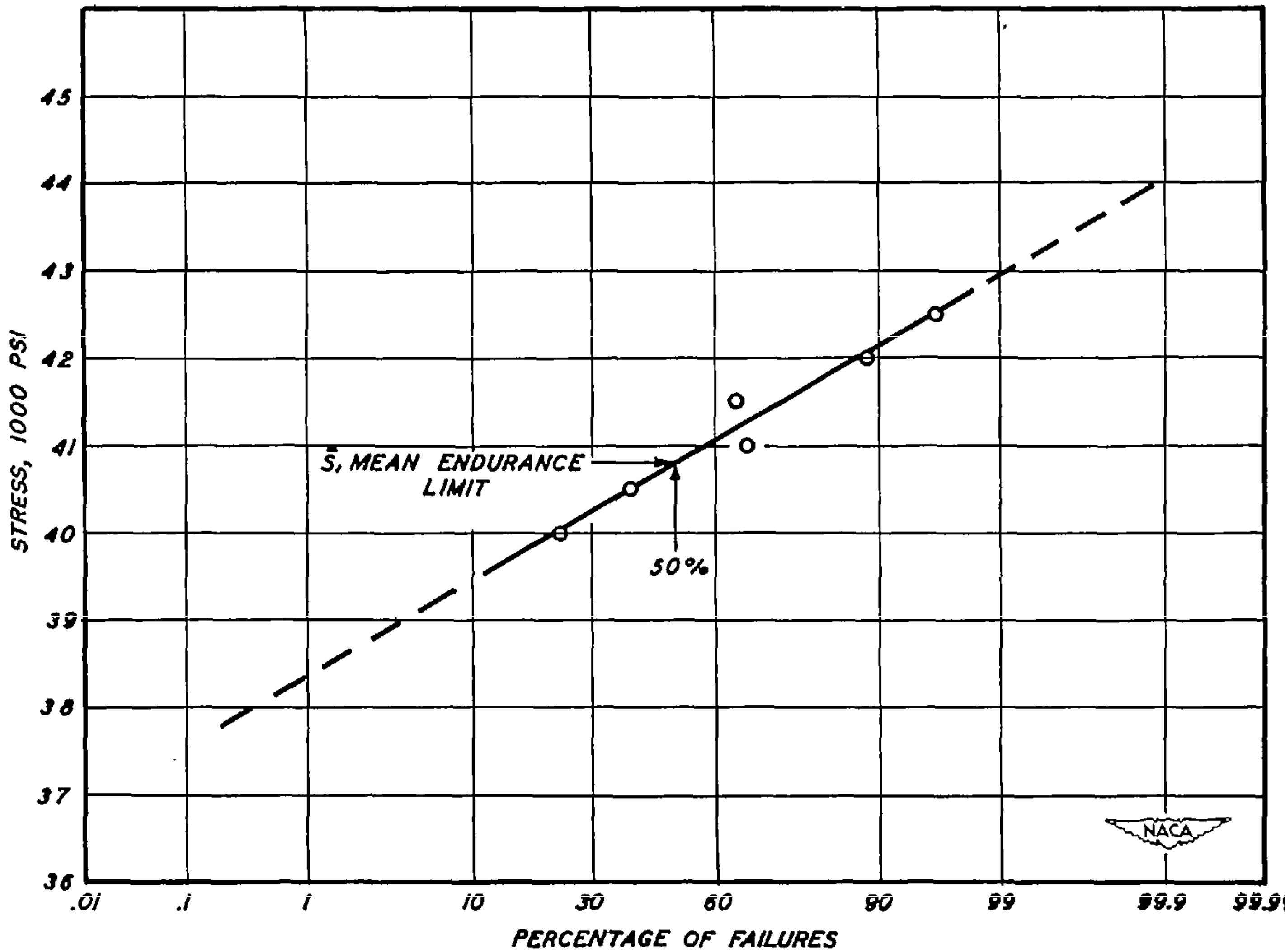
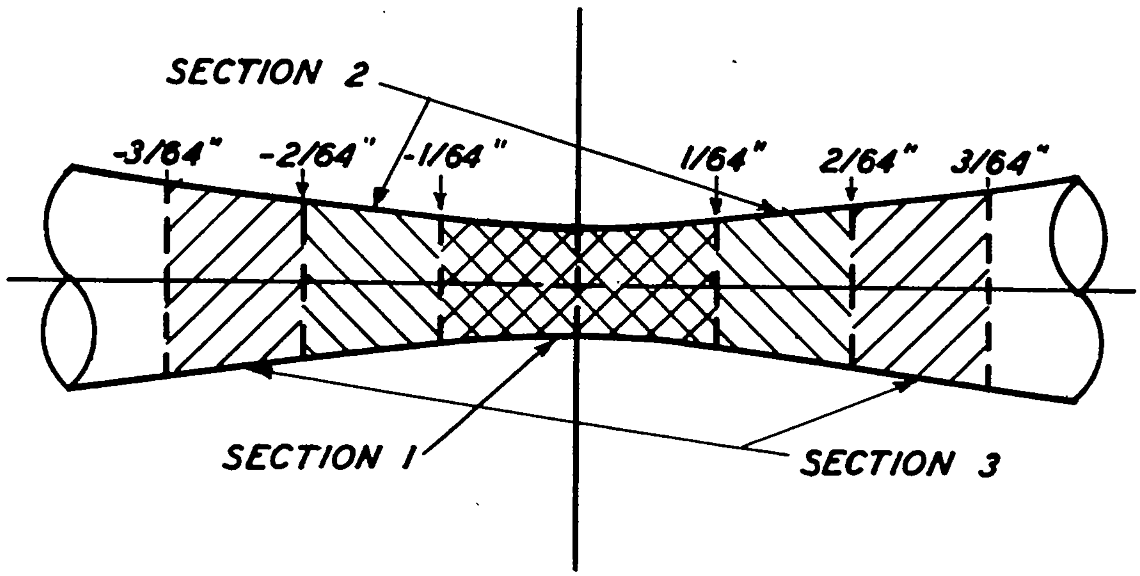


Figure 12.- Mortality curve for endurance range of SAE 1050 in probability coordinates.



SECTION	POSITION
1	0 ± 1/64"
2	1/64" TO 2/64", -1/64" TO -2/64"
3	2/64" TO 3/64", -2/64" TO -3/64"
.	.
.	.
.	.
10	9/64" TO 10/64", -9/64" TO -10/64"

Figure 13.- Location of sections along specimen gage length.

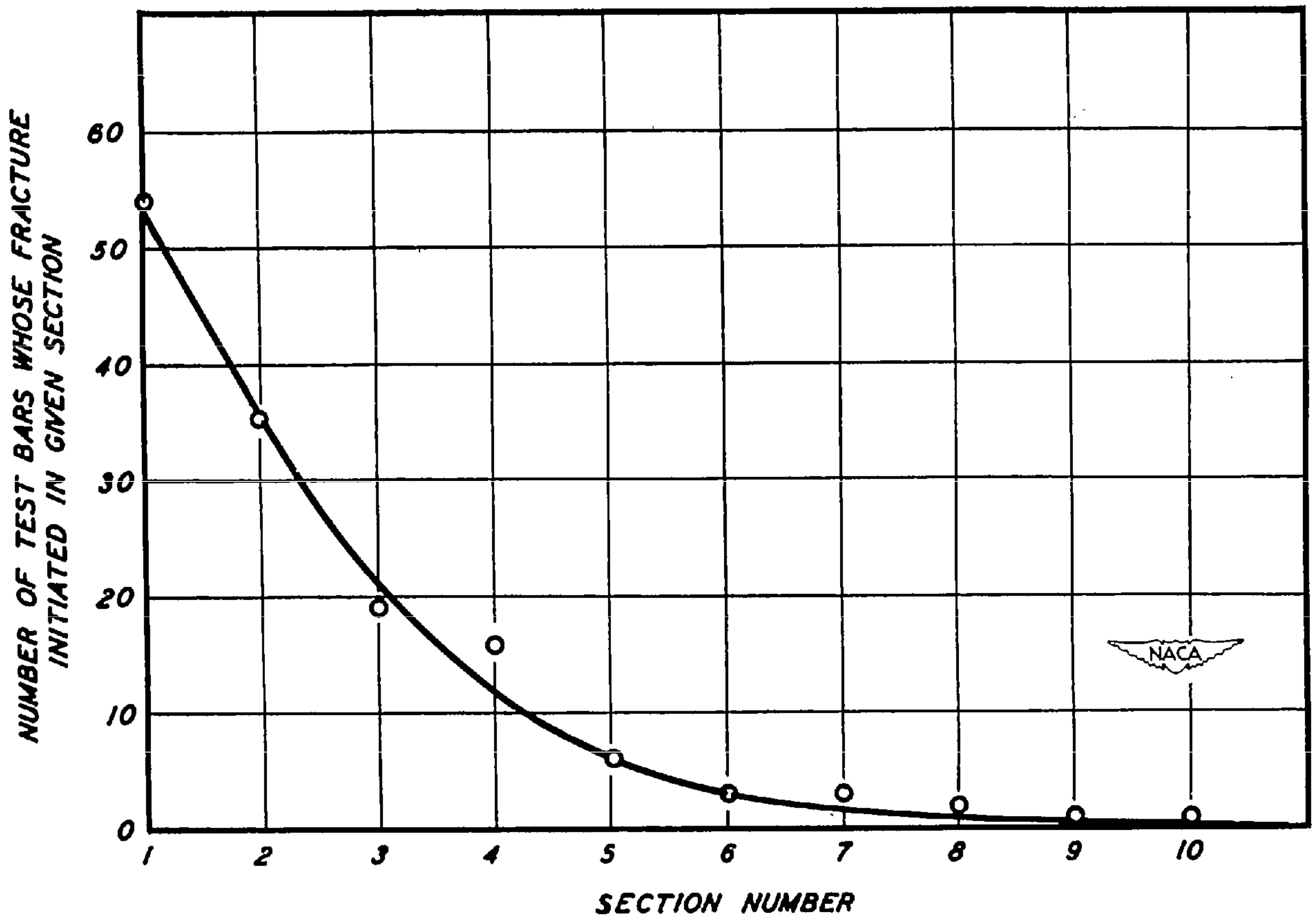


Figure 14.- Frequency distribution of failure location in linear coordinates for SAE 1050.

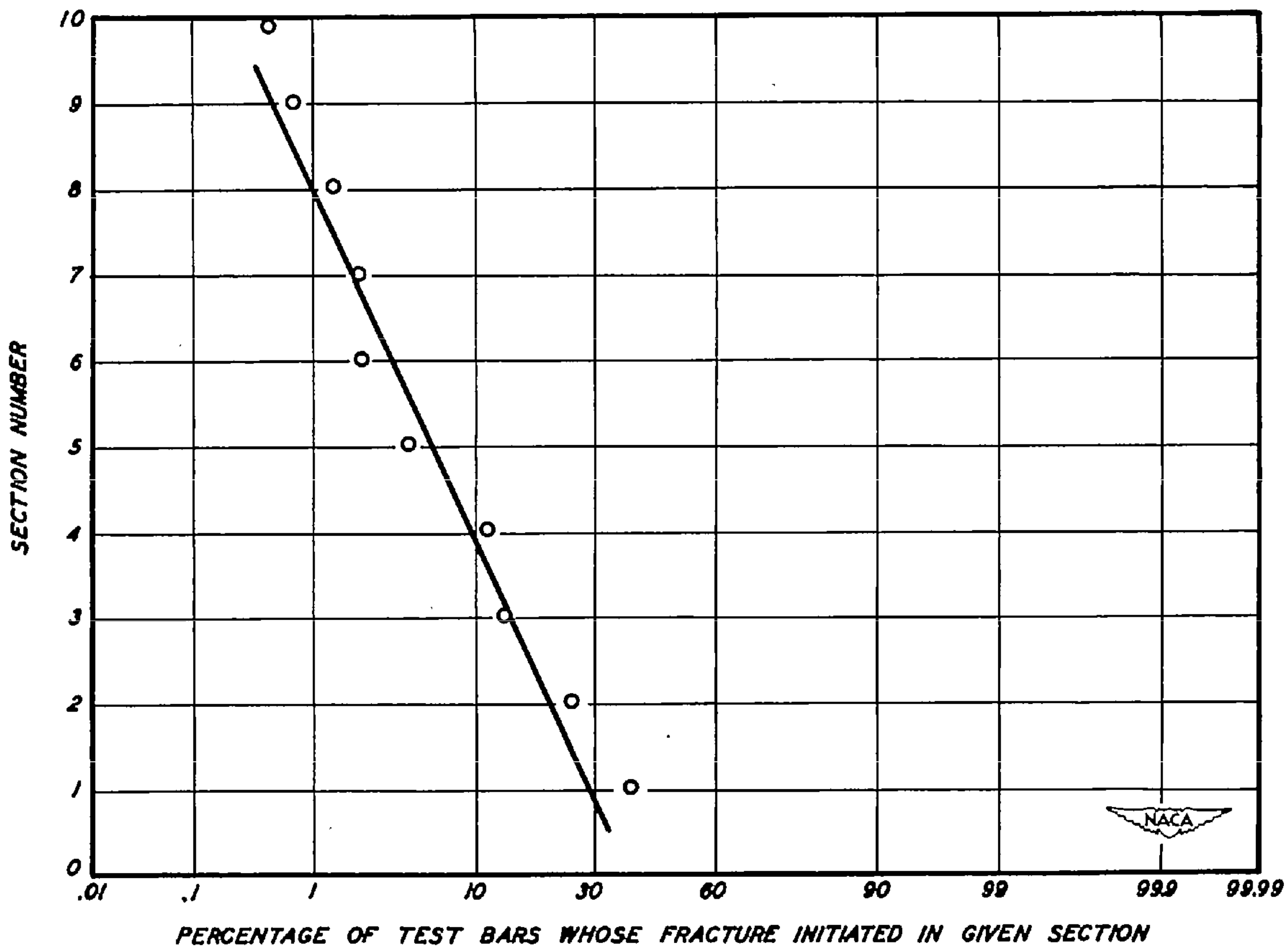


Figure 15.- Frequency distribution of failure location in probability coordinates for SAE 1050.

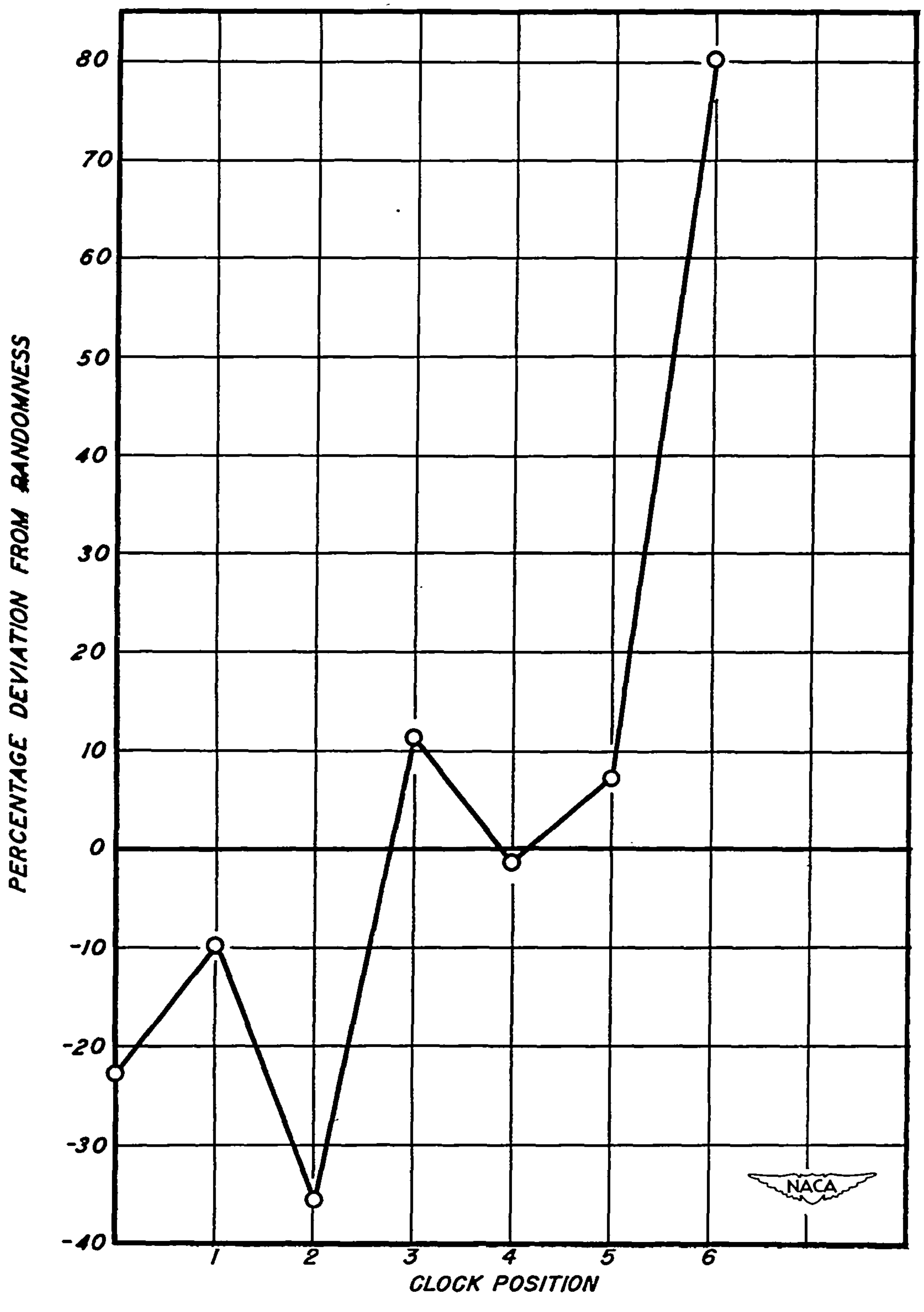


Figure 16.- Percentage deviation of observed number of failures in circumferential positions from theoretical number for random occurrence for SAE 1050.

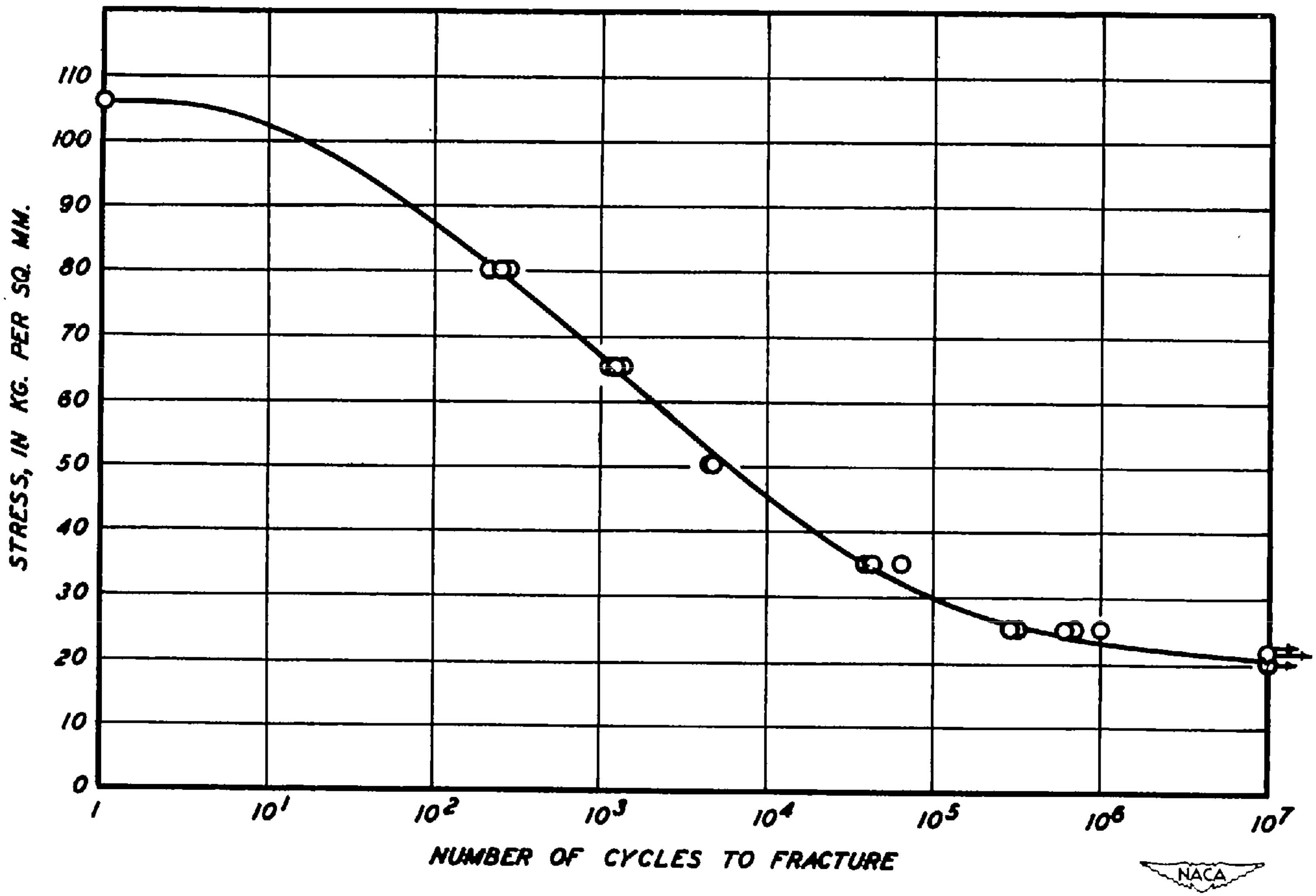


Figure 17.- Fatigue data for chromium-molybdenum steel. (Figure taken reference 21.)

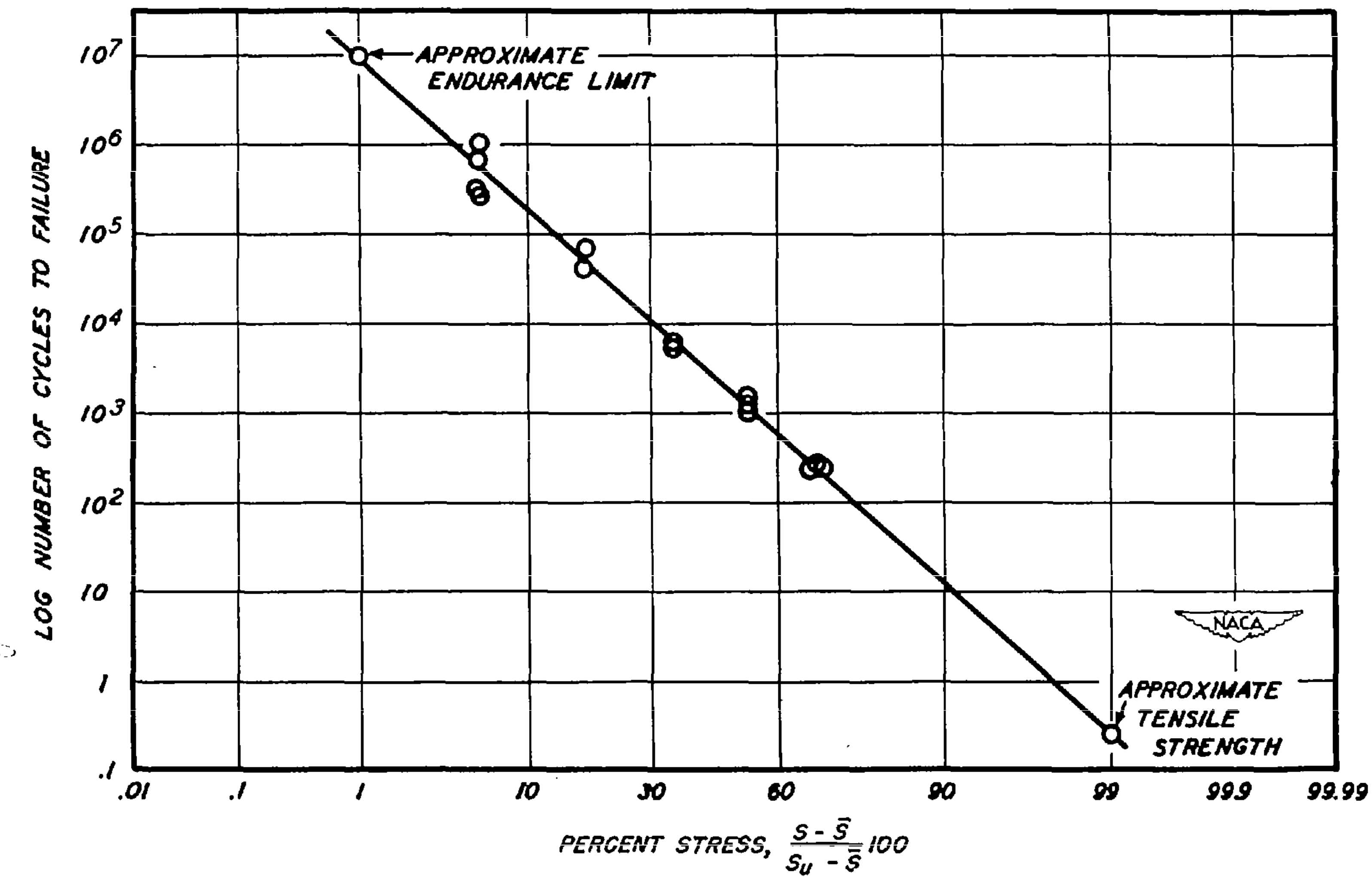


Figure 18.- Proposed new method of plotting fatigue data. (Data from fig. 17.)

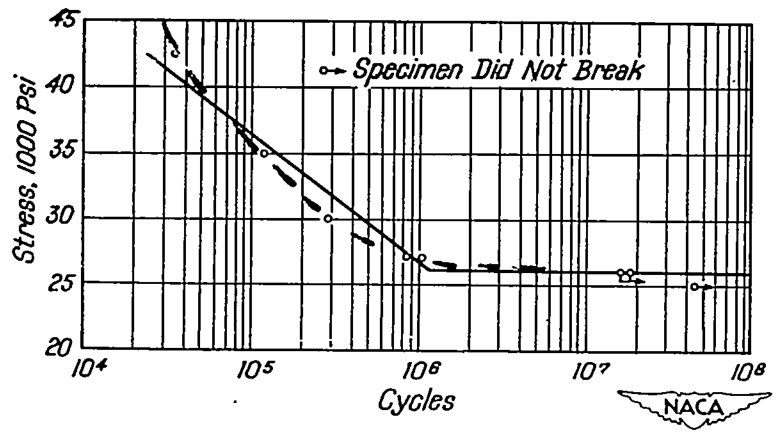


Figure 19.- Fatigue data for pure titanium. (Figure taken from reference 24.)

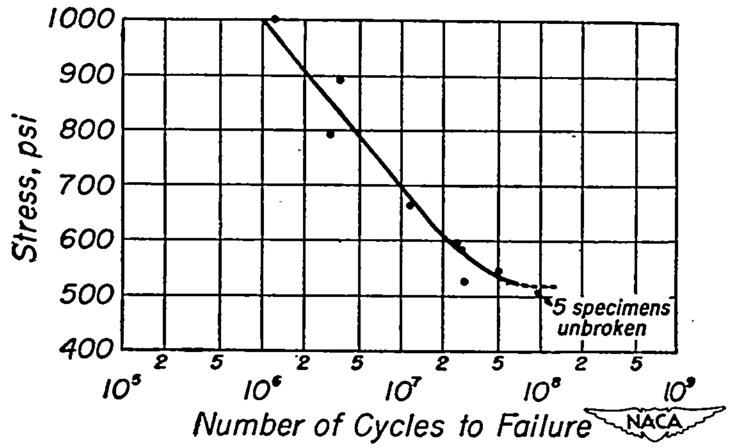


Figure 20.- Fatigue data for pure lead. (Figure taken from reference 25.)

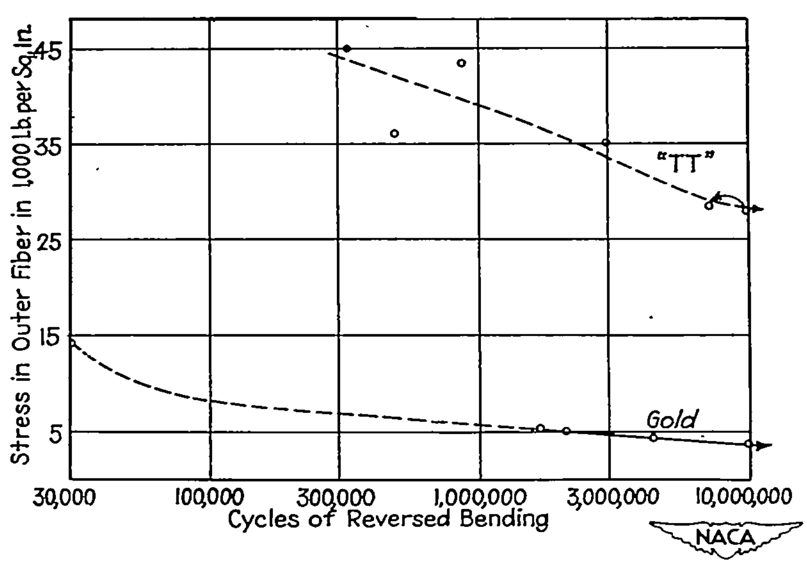


Figure 21.- Fatigue data for pure gold and gold alloy. (Figure taken from reference 26.)

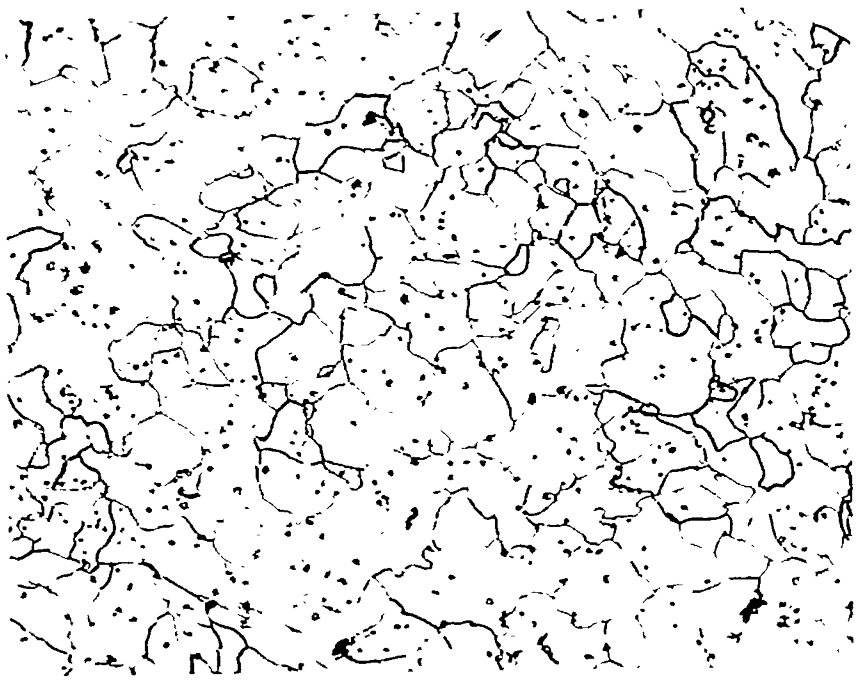


Figure 22.- Microstructure of annealed Armco iron. 100X; picral etch.

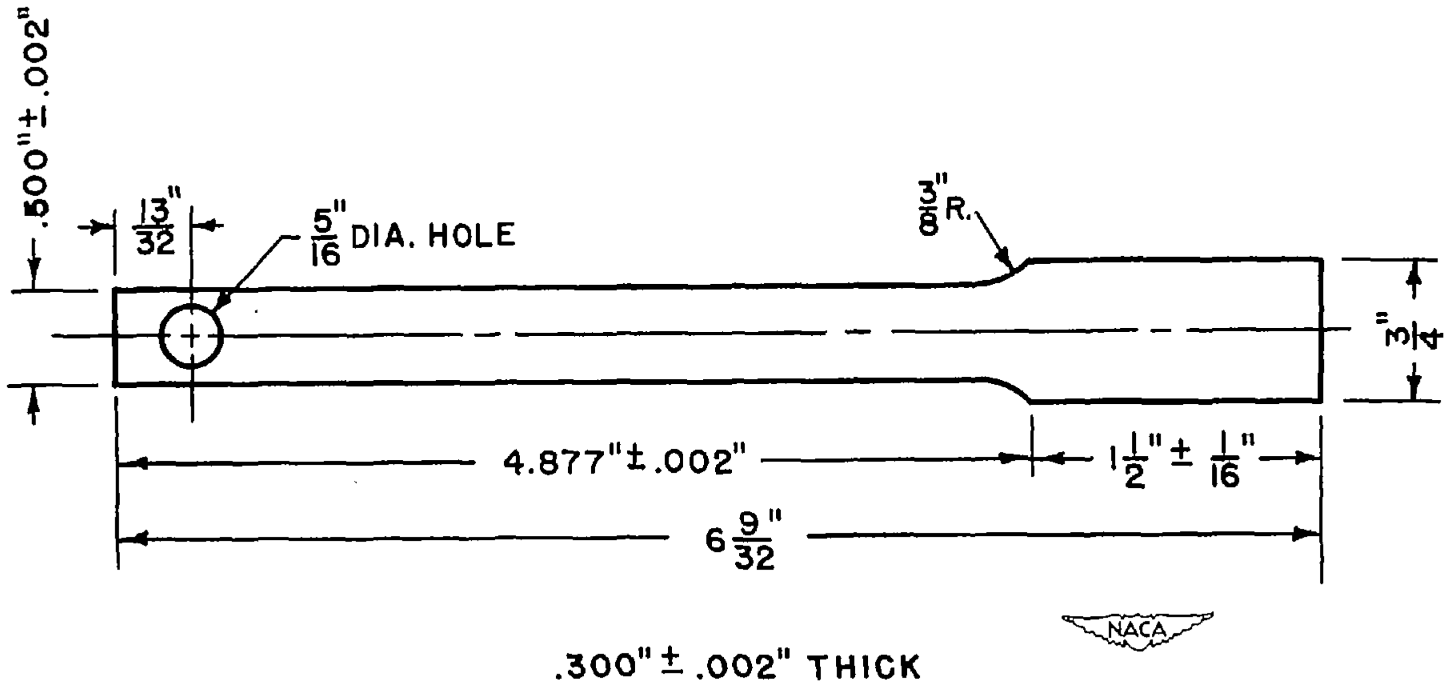


Figure 23.- Fatigue specimen for pneumatic machine.

Figure 24.- Microstructure of SAE 4340, quenched and tempered 16 hours at 525° C. 1000X; nital etch.

Figure 25.- Microstructure of SAE 4340, quenched and spheroidized 16 hours at 600° C. 1000X; nital etch.

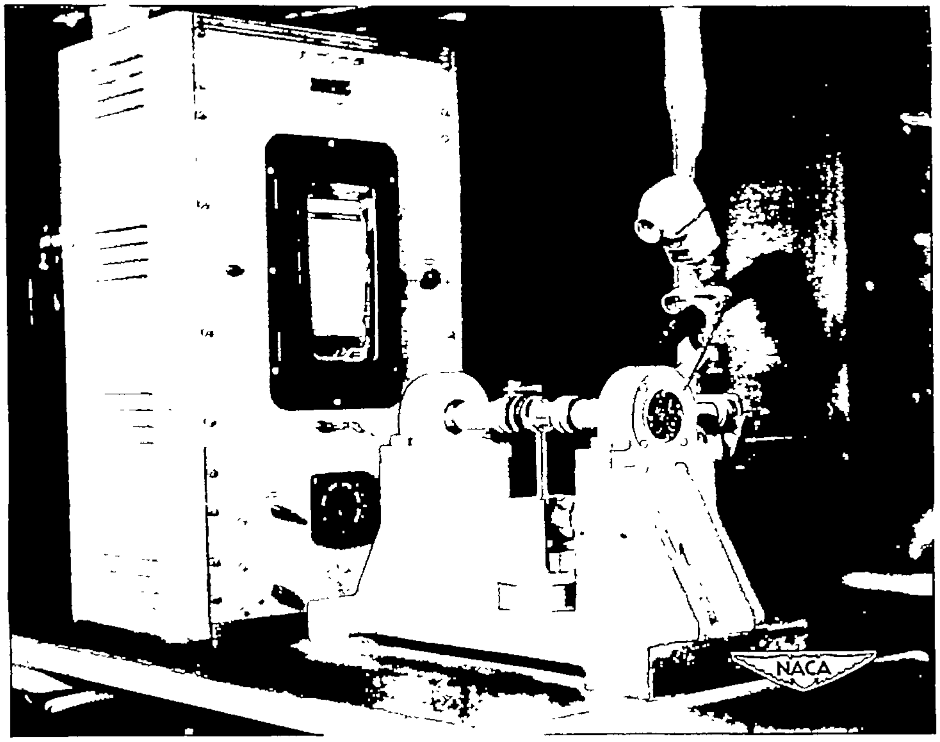


Figure 26.- Pneumatic fatigue machine.

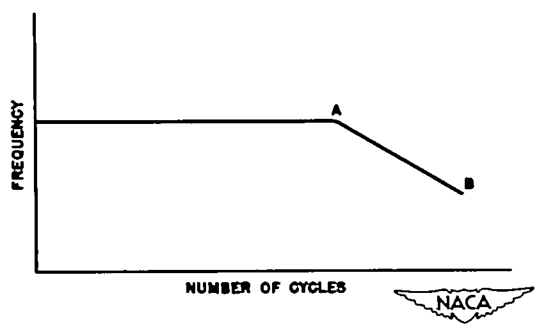


Figure 27.- Frequency of vibration behavior during fatigue test.

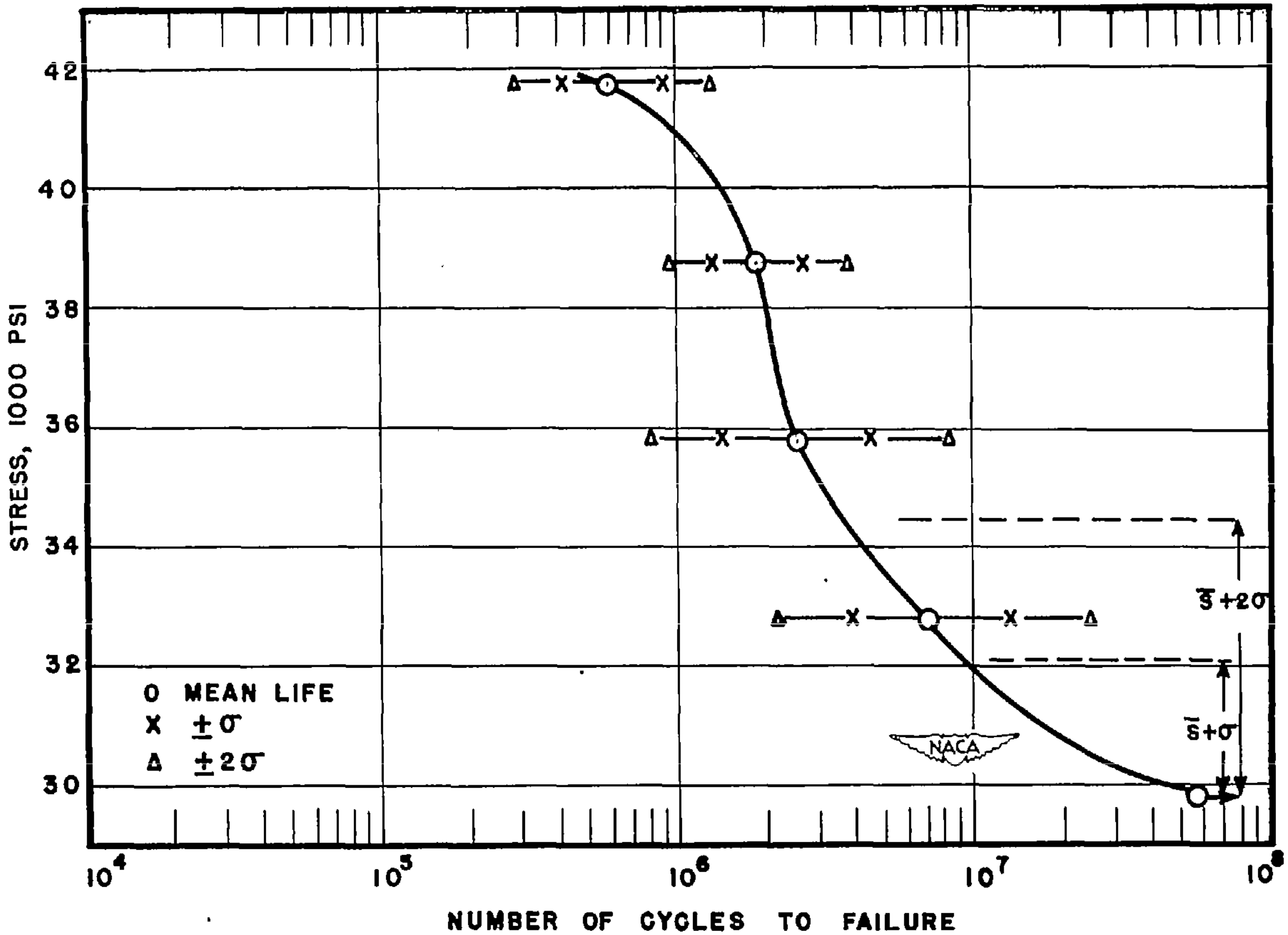


Figure 28.- Statistical variation in fatigue life and endurance limit of annealed Armco iron.

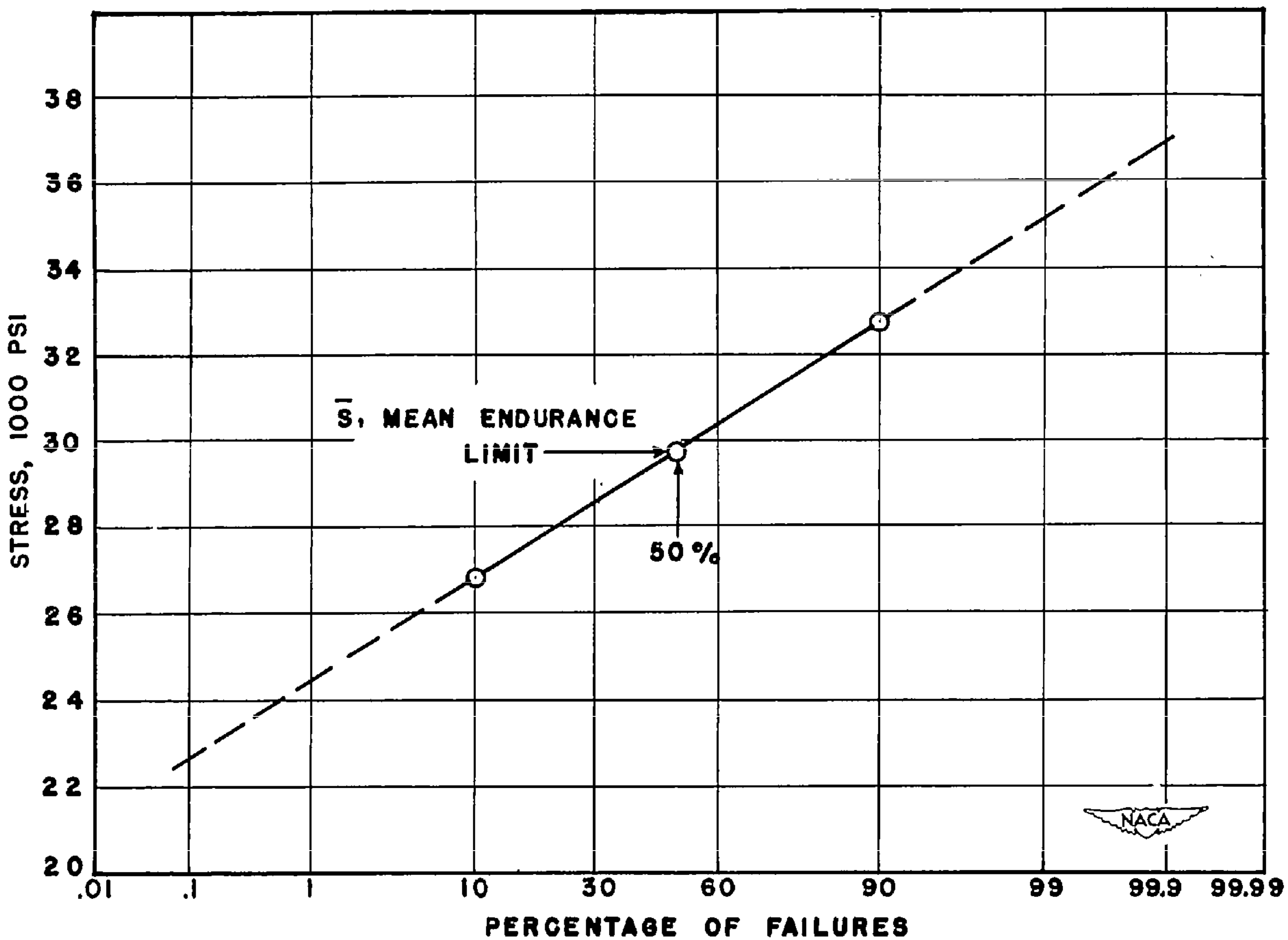


Figure 29.- Mortality curve for endurance range of annealed Armco iron.

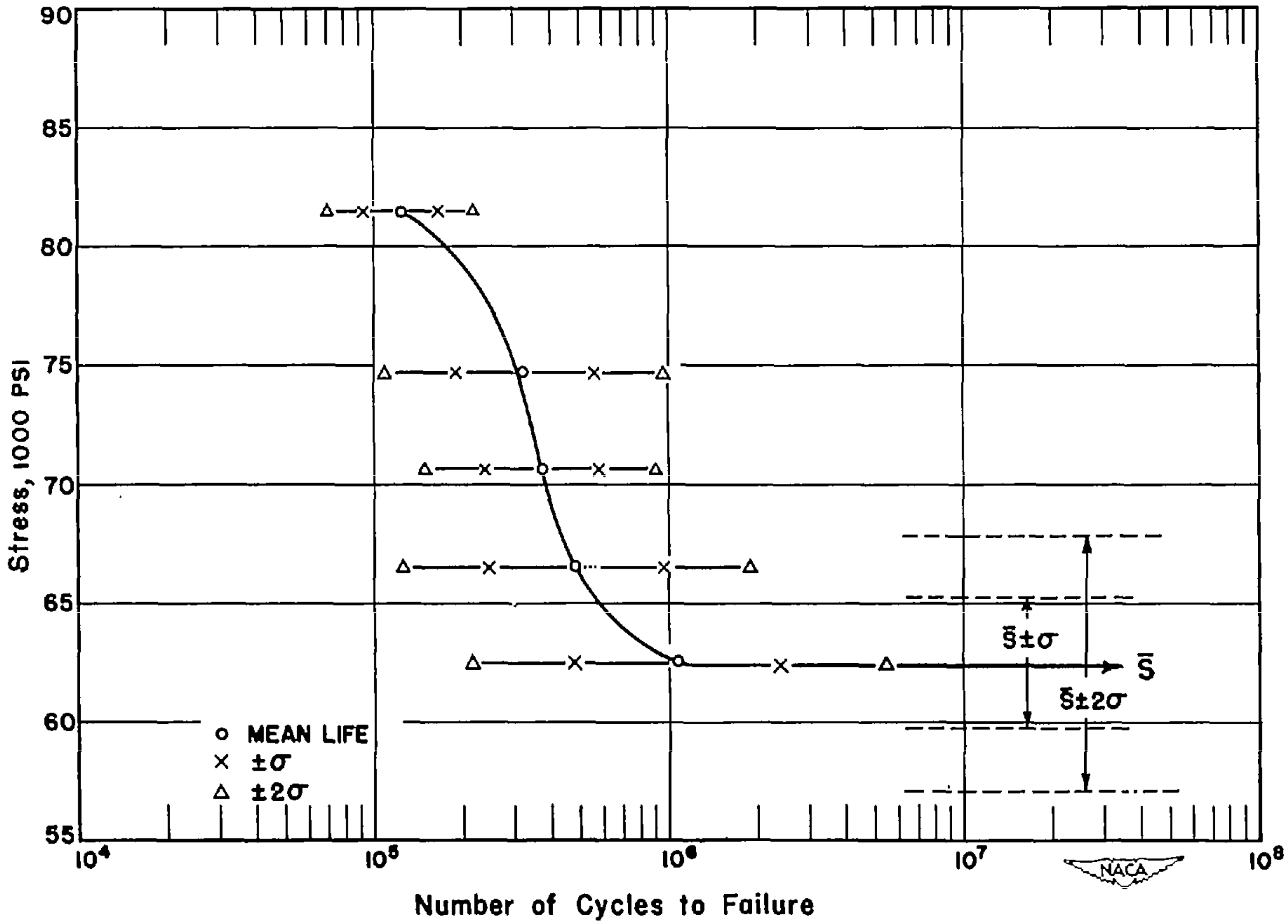


Figure 30.- Statistical variation in fatigue life and endurance limit of quenched and spheroidized SAE 4340.

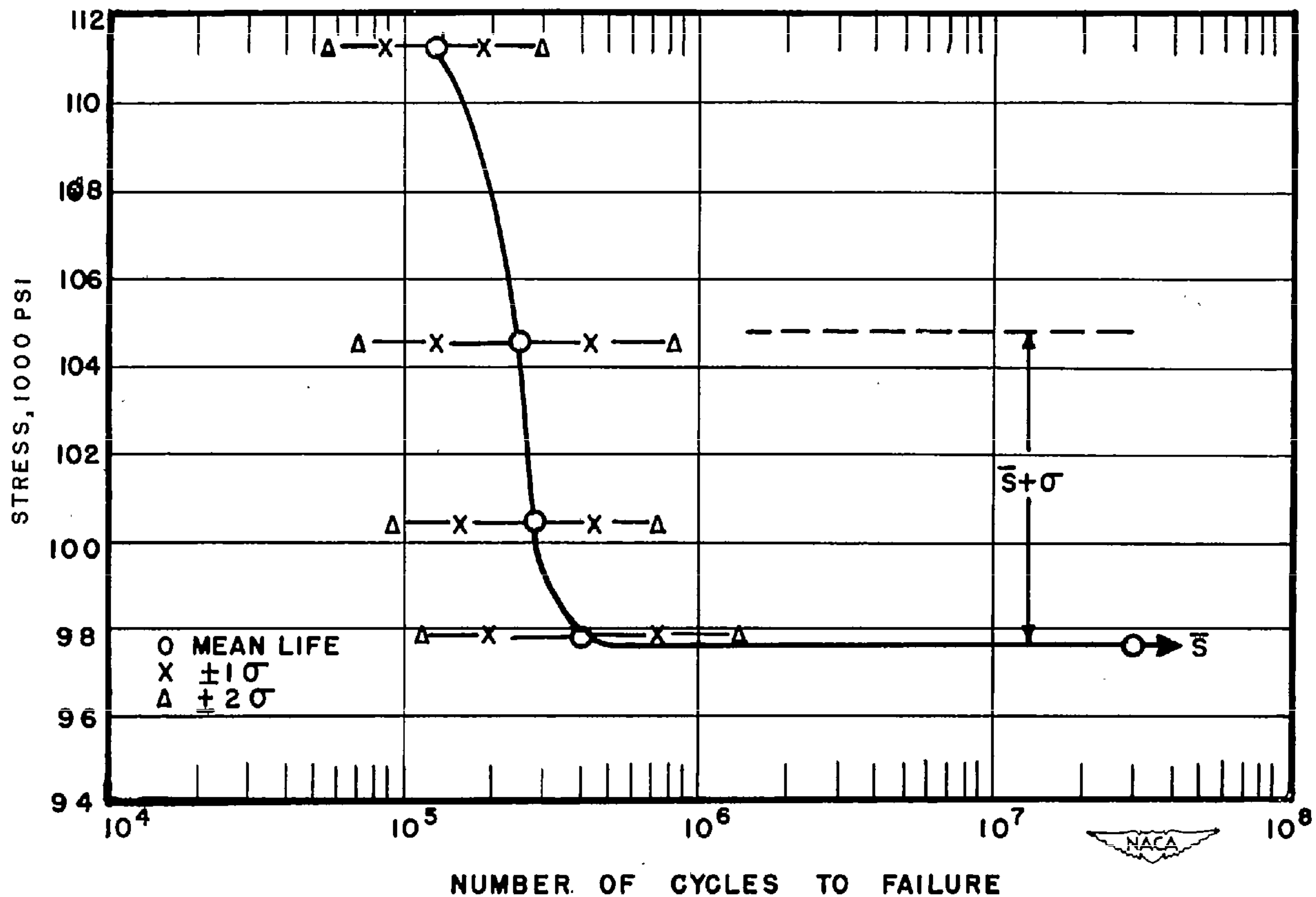


Figure 31.- Statistical variation in fatigue life and endurance limit for quenched and tempered SAE 4340.

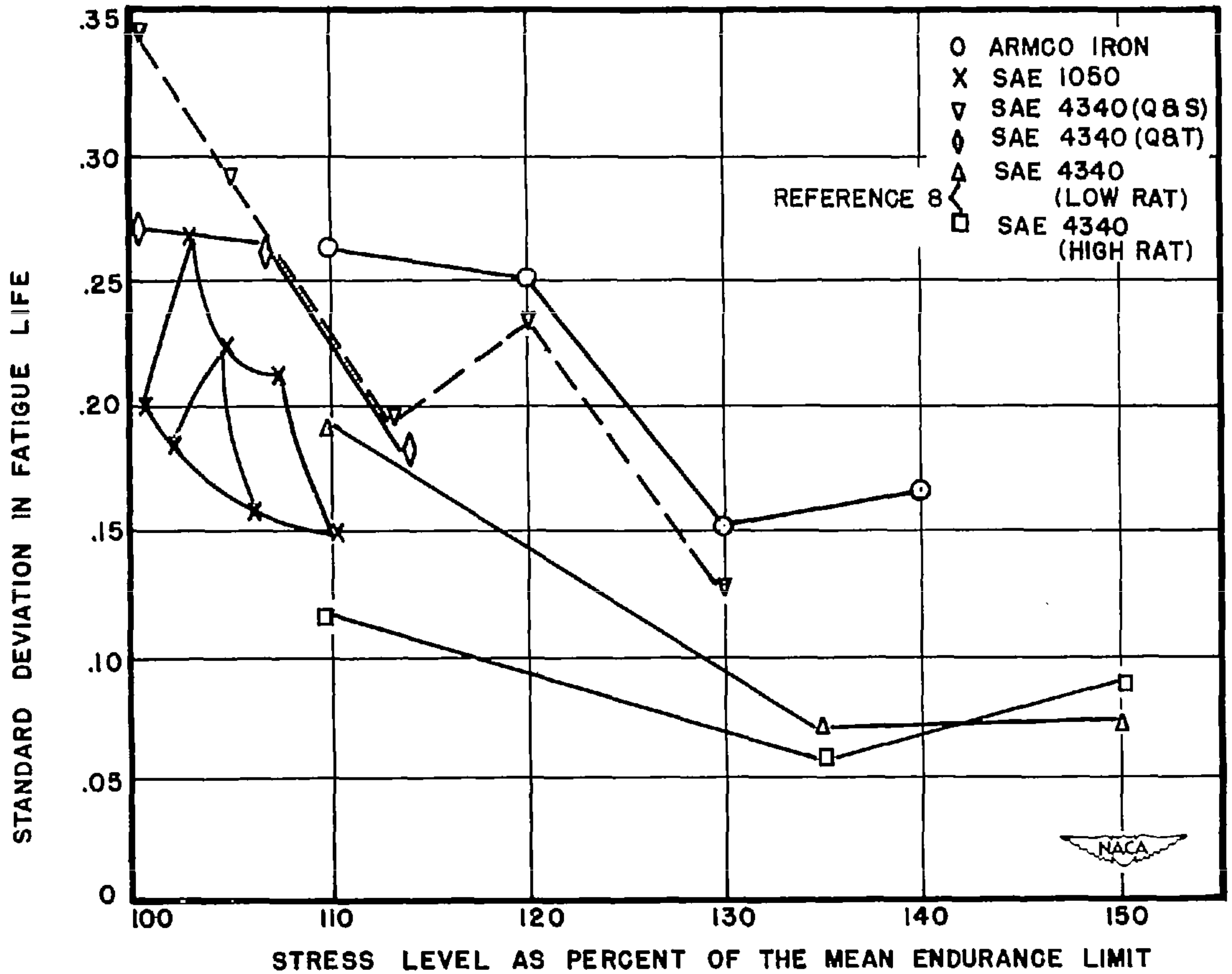


Figure 32.- Summary plot of statistics of fracture curves.

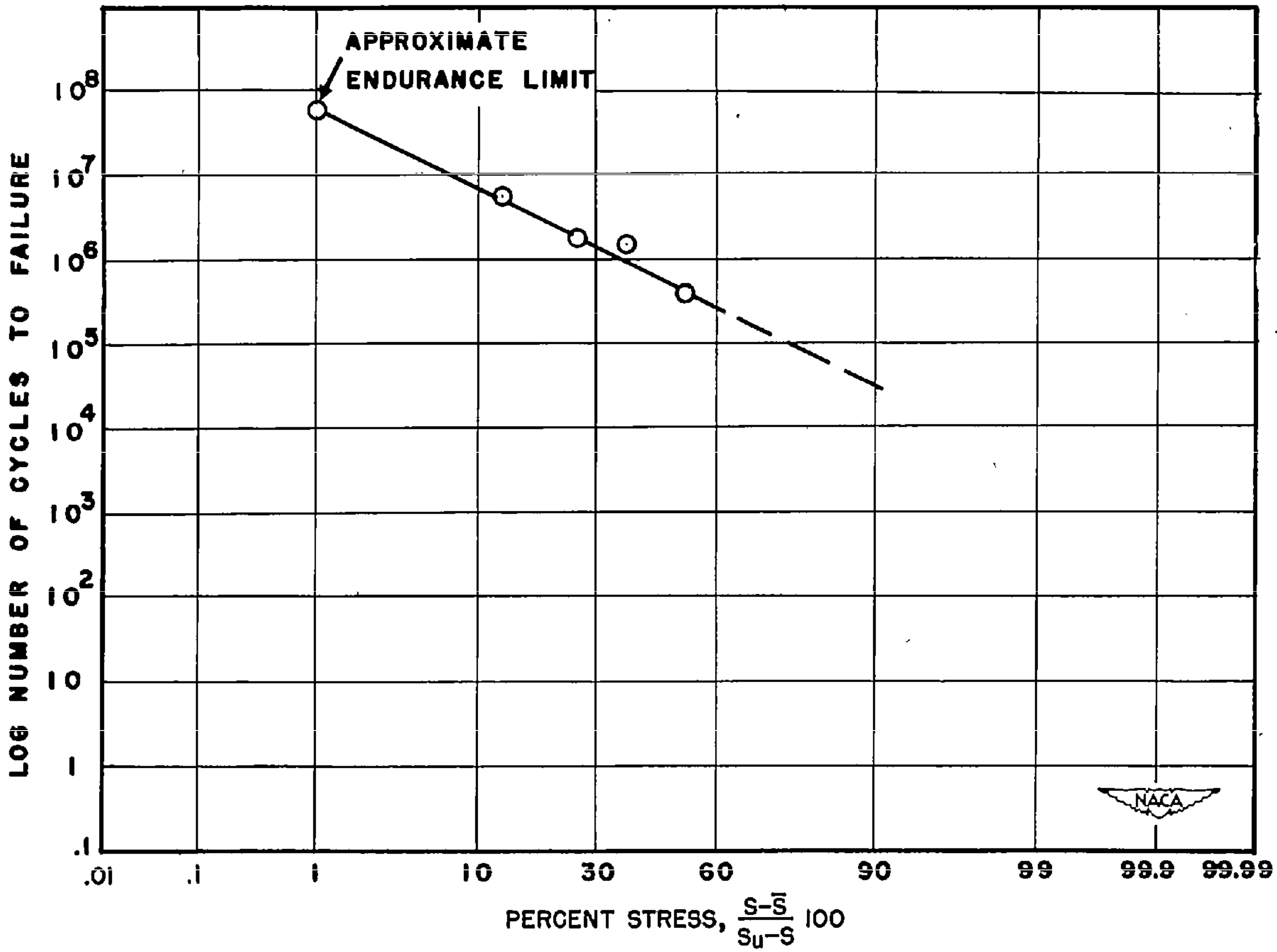


Figure 33.- S-N plot for annealed Armco iron by proposed method.

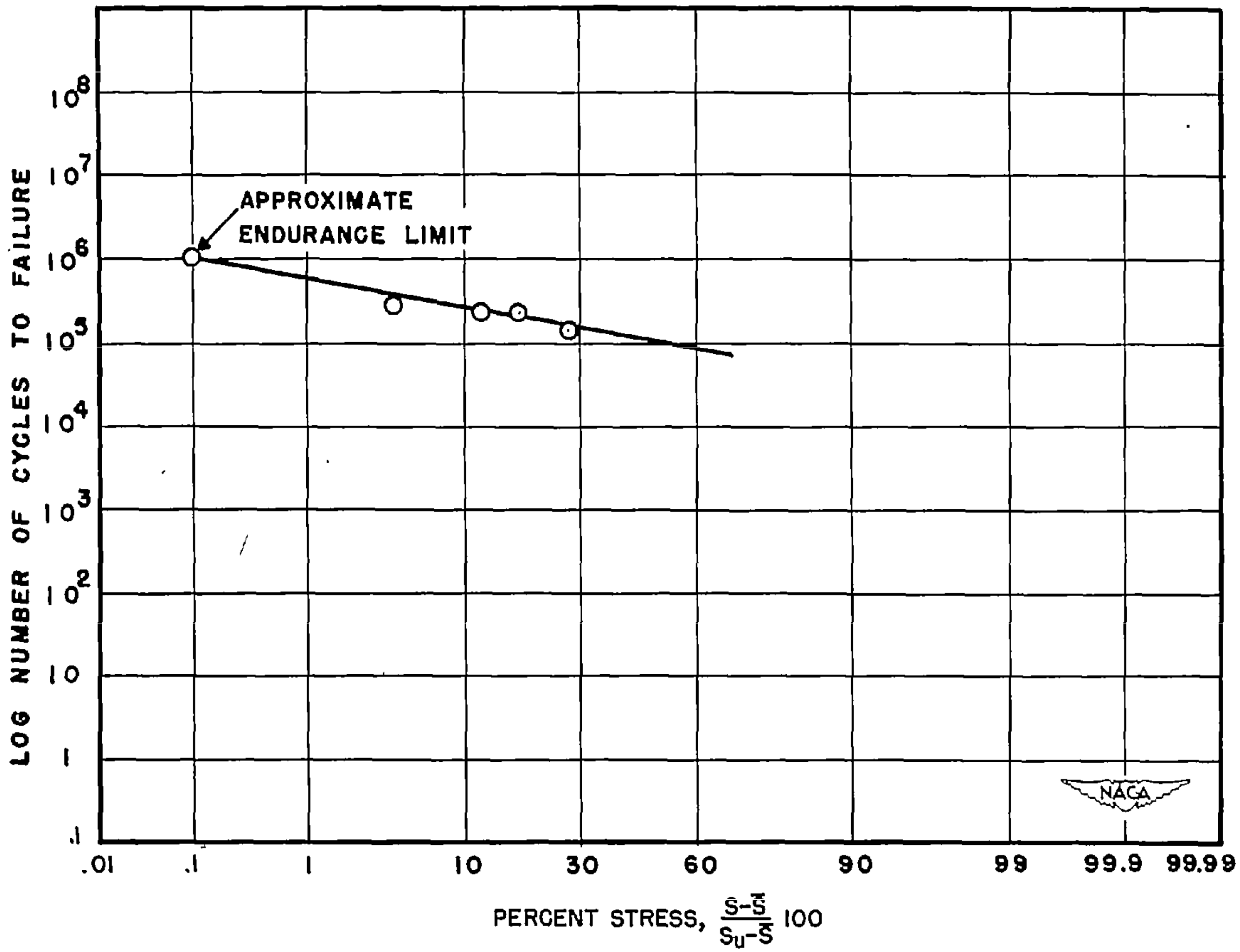


Figure 34.- S-N plot for quenched and spheroidized SAE 4340 by proposed method.

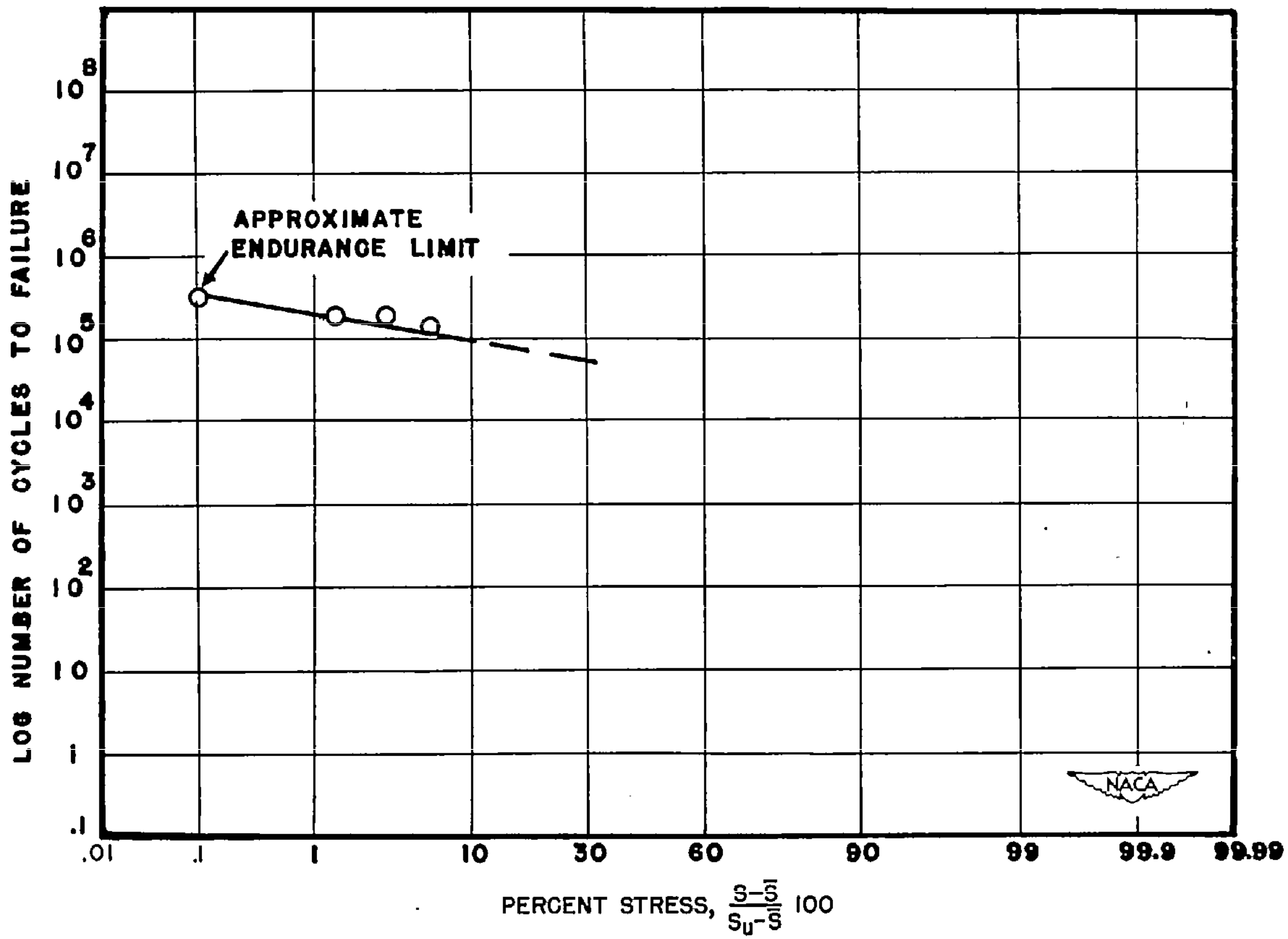


Figure 35.- S-N plot for quenched and tempered SAE 4340 by proposed method.

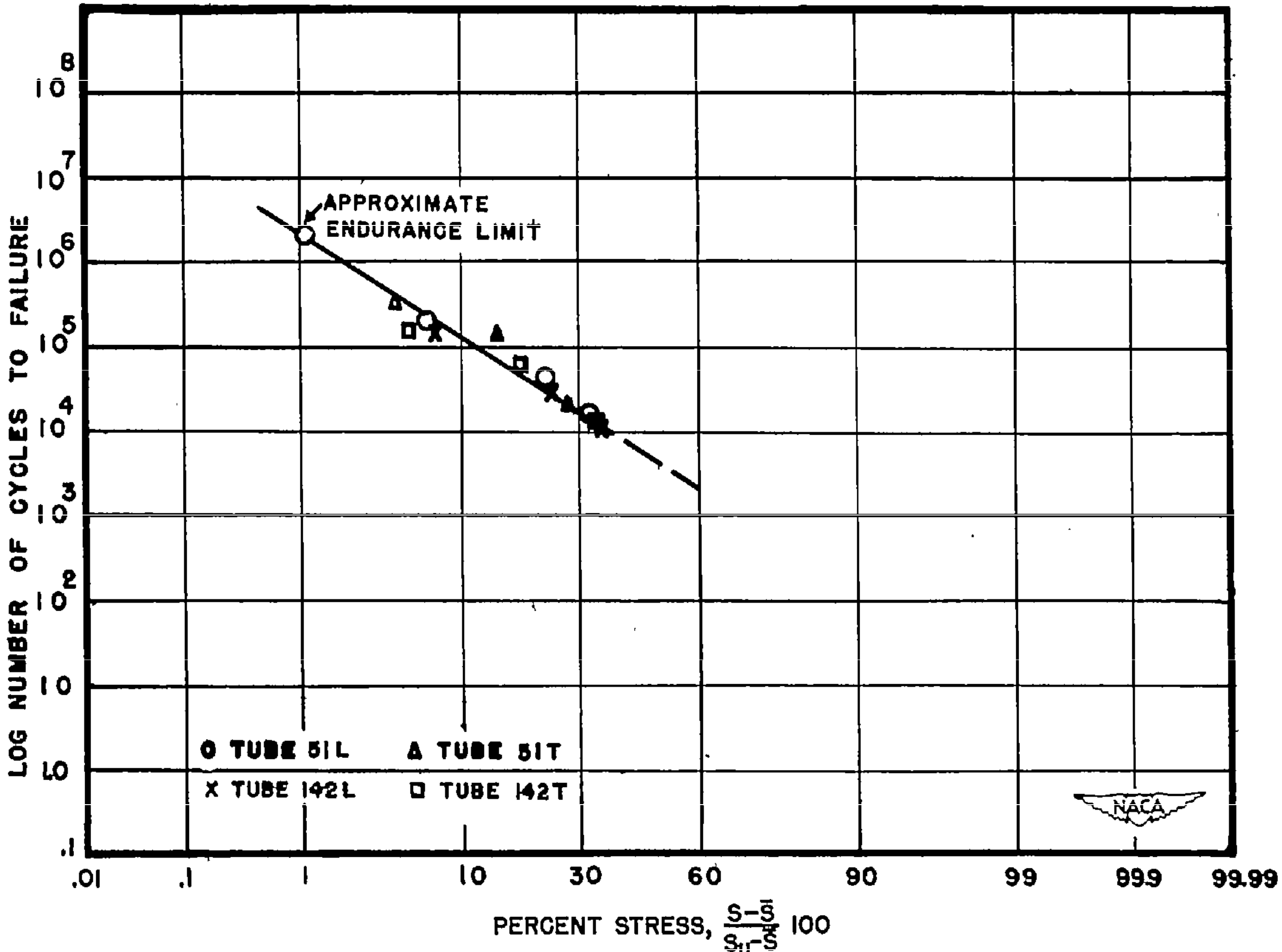


Figure 36.- S-N plot for SAE 4340 (Ransom's data from reference 8) by proposed method.

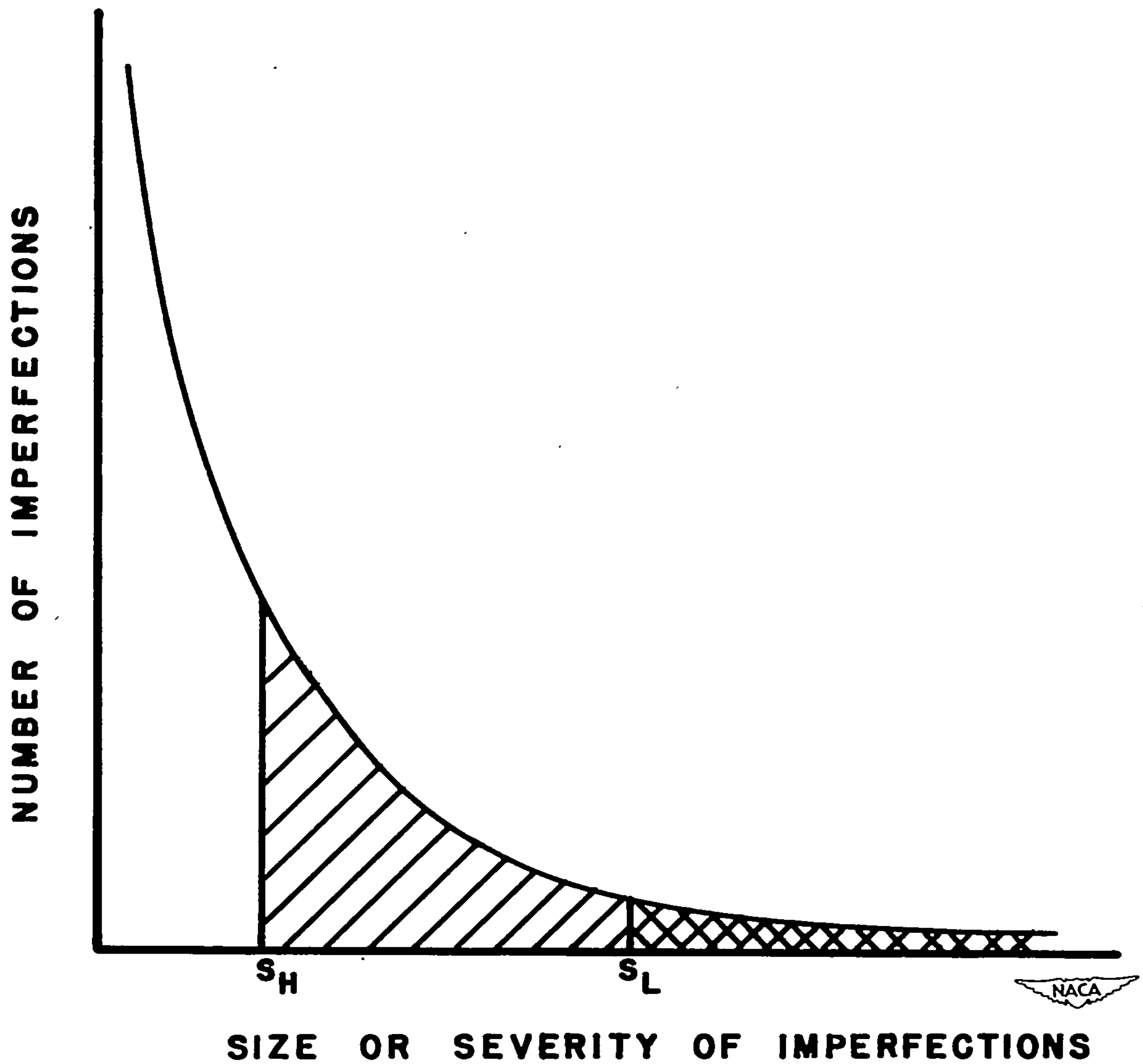


Figure 37.- Frequency distribution of effective imperfections in metal. Shaded areas indicate total number of imperfections available at given stress. S_H and S_L are minimum size or severity of effective imperfections at high and low stress, respectively.

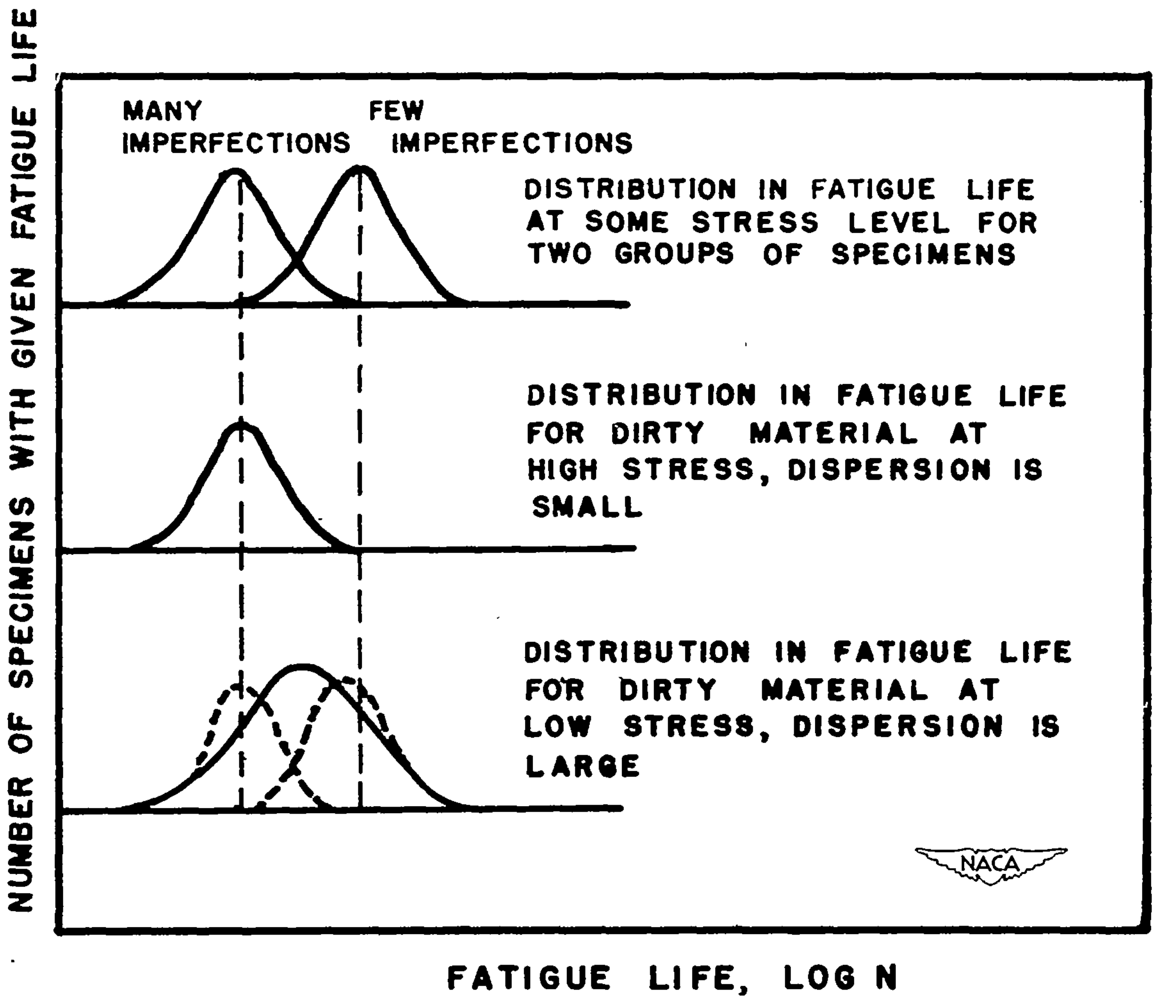


Figure 38.- Distributions in fatigue life of dirty metal as affected by number of imperfections and stress level.

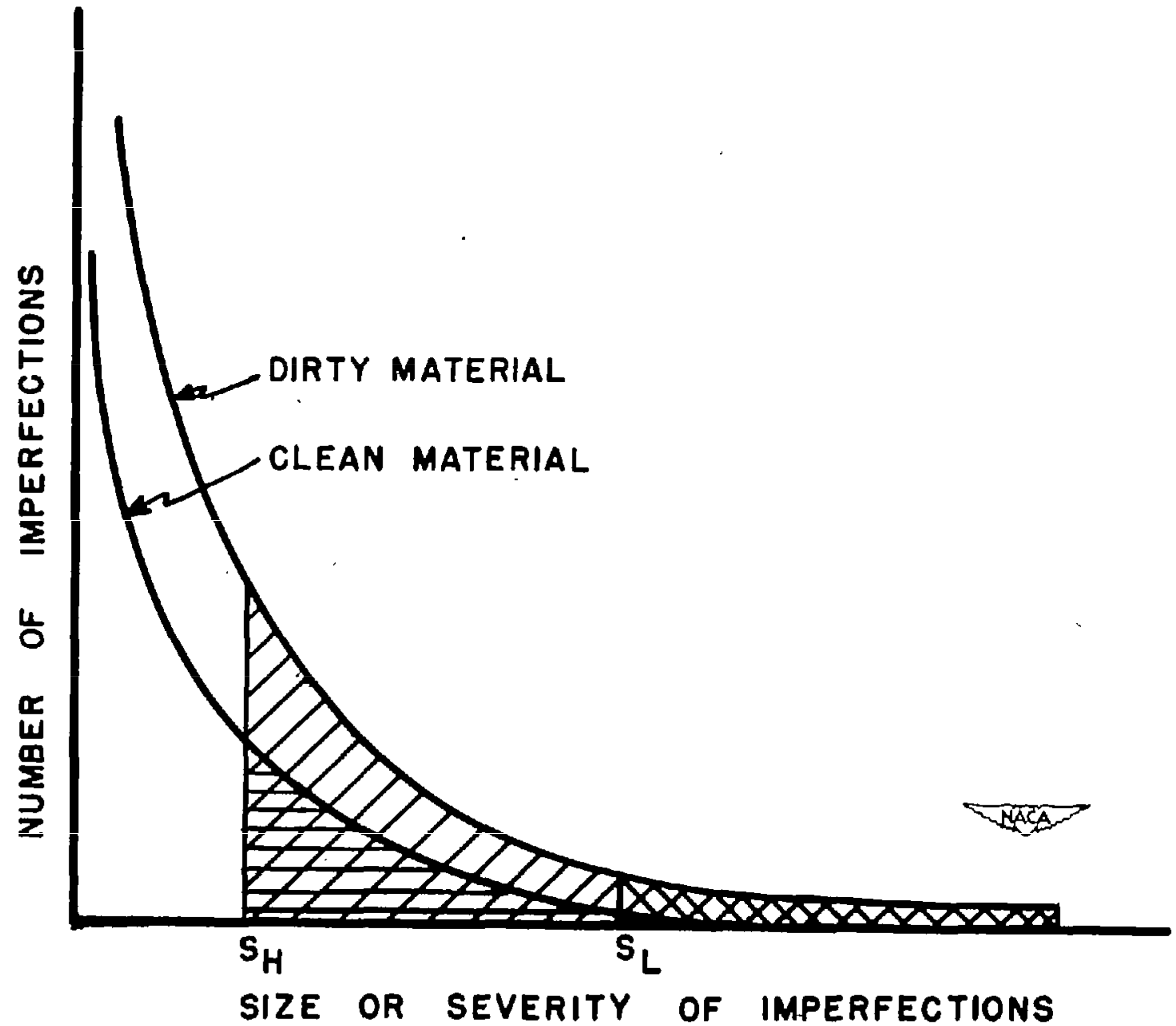


Figure 39.- Frequency distribution of effective imperfections in clean and dirty metal. Shaded areas indicate total number of imperfections available at given stress. S_H and S_L are minimum size or severity of effective imperfections at high and low stress, respectively.

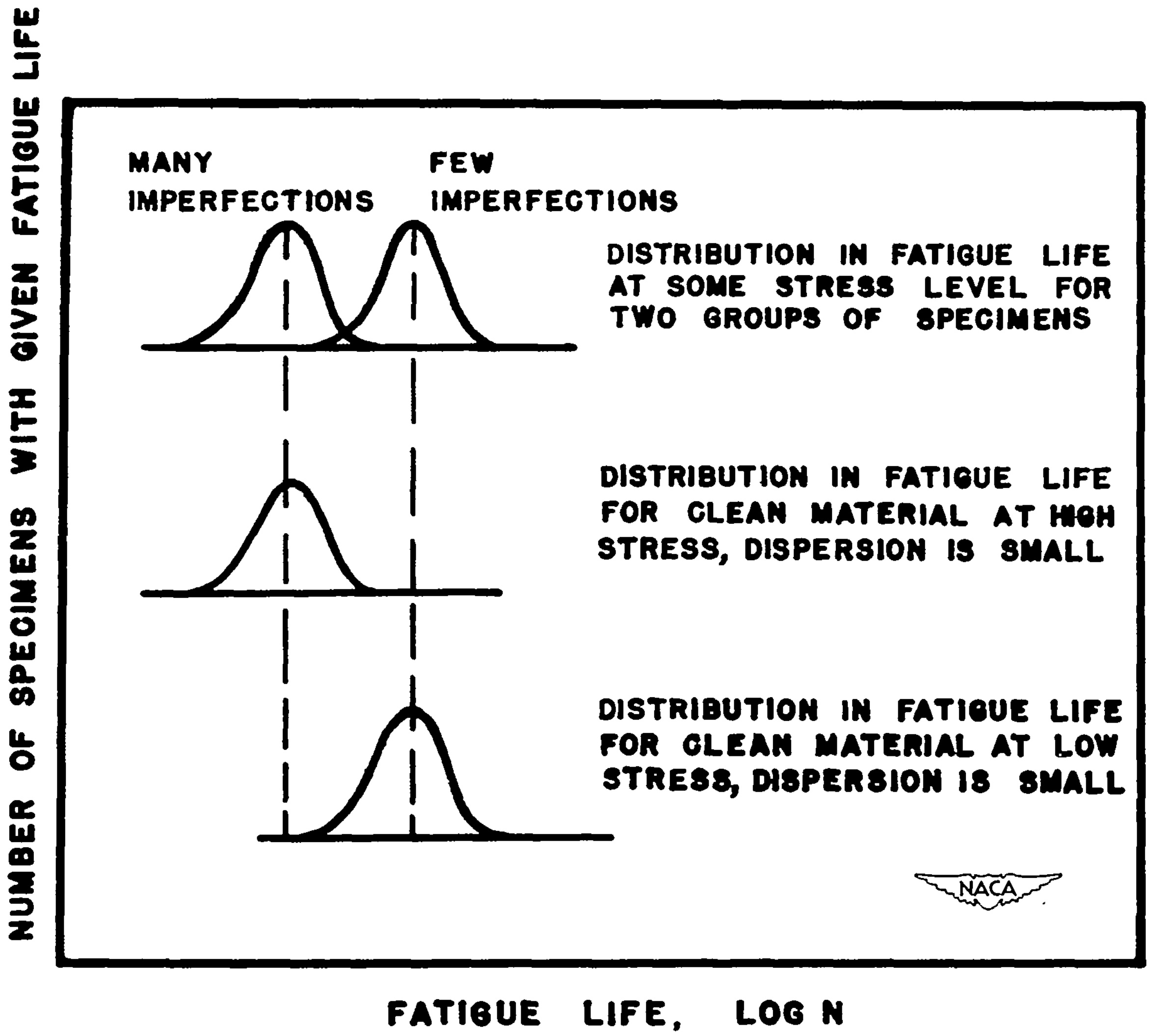


Figure 40.- Distributions in fatigue life of clean metal as affected by number of imperfections and stress level.

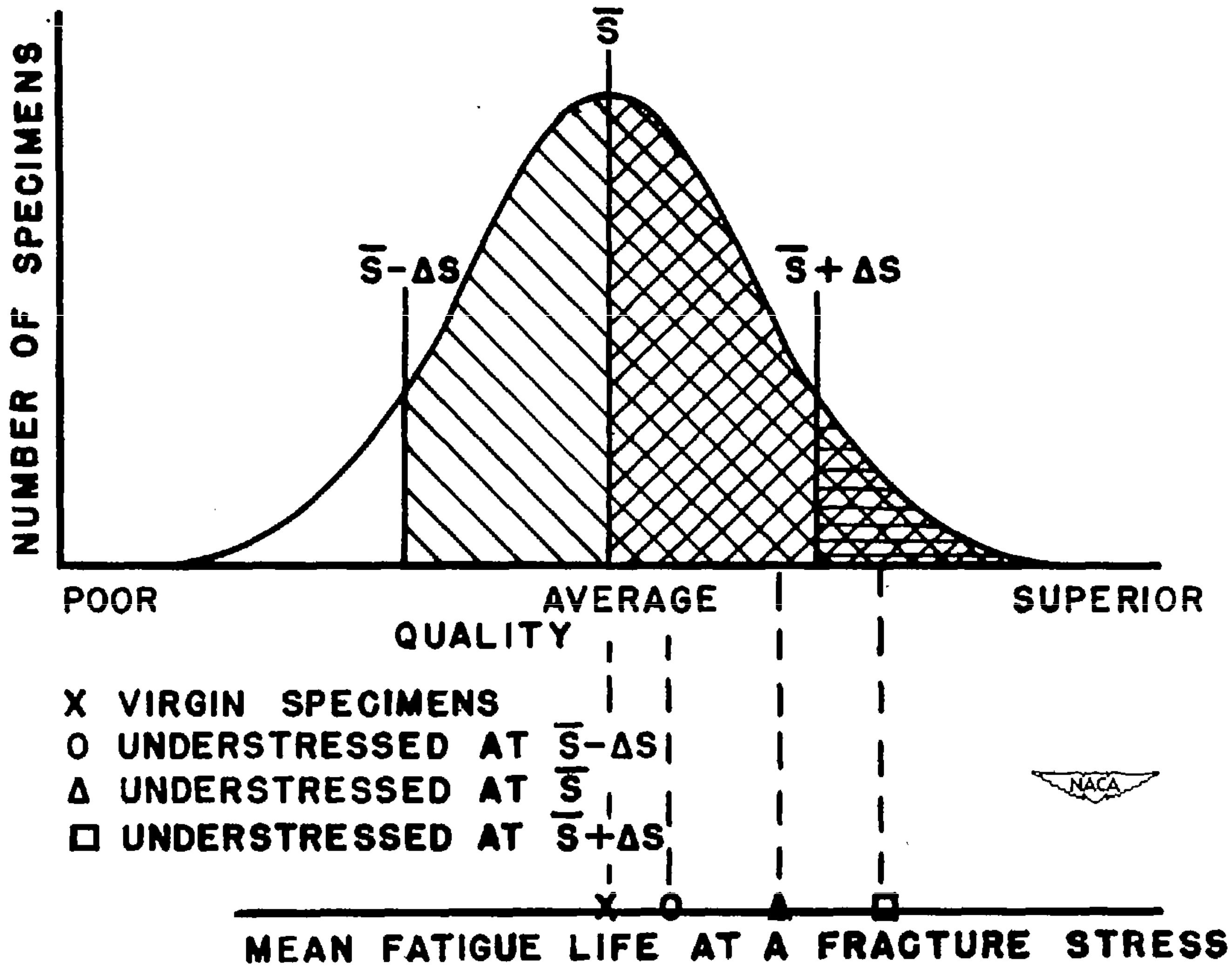


Figure 41.- Distribution in quality of virgin specimens and mean fatigue life of understressed specimens predicted on basis of selectivity.